Non-commutative Polynomials of Independent Gaussian Random Matrices. The Real and Symplectic Cases.

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Abstract

In [HT2] Haagerup and Thorbjørnsen prove the following extension of Voiculescu's random matrix model (cf. [V2, Theorem 2.2]): For each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_r^{(n)}$ be a system of r independent complex self-adjoint random matrices from the class $\mathrm{SGRM}(n, \frac{1}{n})$, and let $x_1, \ldots, x_r$ be a semicircular system in a C^*-probability space. Then for any polynomial p in r non-commuting variables the convergence

$$\lim_{n \to \infty} \|p(X_1^{(n)}, \ldots, X_r^{(n)})\| = \|p(x_1, \ldots, x_r)\|$$

holds almost surely. We generalize this result to sets of independent Gaussian random matrices with real or symplectic entries (the GOE- and the GSE-ensembles) and random matrix ensembles related to these.

1 Introduction.

Throughout this paper we let $(\Omega, \mathcal{F}, P)$ denote a fixed probability space, and for each positive integer n and each $\sigma > 0$ we let $\mathrm{SGRM}(n, \sigma^2)$ denote the set of $n \times n$ self-adjoint Gaussian random matrices defined in [HT2]:

(i) $\mathrm{SGRM}(n, \sigma^2)$ is the set of self-adjoint random matrices $X = (X_{ij}) : \Omega \to M_n(\mathbb{C})$ satisfying that $\{X_{ii} | 1 \leq i \leq n\} \cup \{\sqrt{2}\mathrm{Re}X_{ij} | 1 \leq i < j \leq n\} \cup \{\sqrt{2}\mathrm{Im}X_{ij} | 1 \leq i < j \leq n\}$ is a set of n^2 i.i.d. random variables with distribution $N(0, \sigma^2)$.

We shall also consider the following related random matrix ensembles:

(ii) $\mathrm{GRM}(n, \sigma^2)$, which was defined in [HT2] as well, is the set of random matrices $Y : \Omega \to M_n(\mathbb{C})$ satisfying that the real and the imaginary parts of the entries of Y, $\mathrm{Re}(Y_{ij})$, $\mathrm{Im}(Y_{ij})$, $1 \leq i, j \leq n$, constitute a set of $2n^2$ i.i.d. random variables with distribution $N(0, \frac{1}{2}\sigma^2)$.

(iii) $\mathrm{GRM}^{\mathbb{R}}(n, \sigma^2)$ is the set of random matrices $Y : \Omega \to M_n(\mathbb{R})$ satisfying that the entries of Y, Y_{ij} $1 \leq i, j \leq n$, constitute a set of n^2 i.i.d. random variables with distribution $N(0, \sigma^2)$.

(iv) $\mathrm{GOE}(n, \sigma^2)$ is the set of self-adjoint real random matrices $X : \Omega \to M_n(\mathbb{R})$ satisfying that $\{\frac{1}{\sqrt{2}}X_{ii} | 1 \leq i \leq n\} \cup \{X_{ij} | 1 \leq i < j \leq n\}$ is a set of $\frac{1}{2}n(n+1)$ i.i.d. random variables with distribution $N(0, \sigma^2)$.

(v) $\mathrm{GOE}^*(n, \sigma^2)$ is the set of self-adjoint purely imaginary random matrices $X : \Omega \to M_n(i\mathbb{R})$ satisfying that $\{\mathrm{Im}(X_{ij}) | 1 \leq i < j \leq n\}$ is a set of $\frac{1}{2}n(n-1)$ i.i.d. random variables with distribution $N(0, \sigma^2)$.

Note that (up to scaling) $\mathrm{SGRM}(n, \sigma^2)$ is the Gaussian Unitary Ensemble (GUE) from [Me], and $\mathrm{GOE}(n, \sigma^2)$ is the Gaussian Orthogonal Ensemble (GOE) from [Me].

Also note that for every $Y \in \mathrm{GRM}^{\mathbb{R}}(n, \sigma^2)$ we have

$$X_1 = \frac{1}{\sqrt{2}}(Y + Y^*) \in \mathrm{GOE}(n, \sigma^2),$$

and

$$X_2 = \frac{1}{i\sqrt{2}}(Y - Y^*) \in \mathrm{GOE}^*(n, \sigma^2).$$

Moreover, X_1 and X_2 are stochastically independent. Conversely, if $X_1 \in \mathrm{GOE}(n, \sigma^2)$ and $X_2 \in \mathrm{GOE}^*(n, \sigma^2)$ are independent, then

$$Y = \frac{1}{\sqrt{2}}(X_1 + iX_2) \in \mathrm{GRM}^{\mathbb{R}}(n, \sigma^2). \tag{1.1}$$

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Theorem. [HT2, Theorem 7.1] Let $r \in \mathbb{N}$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_r^{(n)}$ be stochastically independent random matrices from $\mathrm{SGRM}(n, \frac{1}{n})$. Furthermore, let $(\mathcal{A}, \tau)$ be a C^*-probability space with τ a faithful state on $\mathcal{A}$, and let $\{x_1, \ldots, x_r\}$ be a semicircular system in $(\mathcal{A}, \tau)$. Then there is a P-null set $N \subseteq \Omega$ such that for any $\omega \in \Omega \setminus N$ and for any polynomial p in r non-commuting variables:

$$\lim_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega))\| = \|p(x_1, \ldots, x_r)\|.$$

We prove similar results for some other classes of self-adjoint Gaussian random matrices:

Theorem A. Let $r, s \in \mathbb{N}_0$ with $r + s \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ be stochastically independent random matrices defined on $(\Omega, \mathcal{F}, P)$ such that $X_1^{(n)}, \ldots, X_r^{(n)} \in$ GOE$(n, \frac{1}{n})$ and $X_{r+1}^{(n)}, \ldots, X_{r+s}^{(n)} \in$ GOE$^*(n, \frac{1}{n})$. Furthermore, let $(\mathcal{A}, \tau)$ be a C^*-probability space with τ a faithful state on $\mathcal{A}$, and let $\{x_1, \ldots, x_{r+s}\}$ be a semicircular system in $(\mathcal{A}, \tau)$. Then there is a P-null set $N \subseteq \Omega$ such that for any $\omega \in \Omega \setminus N$ and for any polynomial p in $r + s$ non-commuting variables:

$$\lim_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))\| = \|p(x_1, \ldots, x_{r+s})\|.$$

In the sections 2 to 5 we concentrate on proving that there is a P-null set $N' \subseteq \Omega$ such that for any non-commutative polynomial p in $r + s$ variables and for every $\omega \in \Omega \setminus N'$,

$$\limsup_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))\| \leq \|p(x_1, \ldots, x_{r+s})\|, \tag{1.2}$$

and in Section 6 we prove that there is a P-null set $N'' \subseteq \Omega$ such that for any non-commutative polynomial p in $r + s$ variables and for every $\omega \in \Omega \setminus N''$,

$$\liminf_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))\| \geq \|p(x_1, \ldots, x_{r+s})\|. \tag{1.3}$$

Clearly, Theorem A follows from (1.2) and (1.3).

The proof of (1.2) follows the lines of [HT2], and we shall at some places leave out the details and refer to that paper instead. However, additional difficulties arise in the GOE/GOE*-case, and this is mainly due to the appearance of a term of order $\frac{1}{n}$ in our "master equation" (cf. Theorem 2.1),

$$\mathbb{E}\Big\{(a_0 - \lambda)H_n(\lambda) + \sum_{j=1}^{r}[a_j H_n(\lambda)a_j^* H_n(\lambda) + a_j^* H_n(\lambda)a_j H_n(\lambda)] + \mathbf{1}_m\Big\} = -\frac{1}{n}R_n(\lambda),$$

where we use the notation introduced in Section 2. The corresponding equation in [HT2] (cf. [HT2, Theorem 3.6]) contains no such term:

$$\mathbb{E}\Big\{(a_0 - \lambda)H_n(\lambda) + \sum_{j=1}^{r}a_j H_n(\lambda)a_j H_n(\lambda) + \mathbf{1}_m\Big\} = 0.$$

A crucial ingredient in the proof given by Haagerup and Thorbjørnsen is the following estimate (cf. [HT2, Theorem 5.7]):

$$\|G_n(\lambda) - G(\lambda)\| = O\Big(\frac{1}{n^2}\Big),$$

Because of the difference mentioned above we get an extra term of order $\frac{1}{n}$ (cf. Theorem 4.4):

$$\Big\|G_n(\lambda) - G(\lambda) - \frac{1}{n}L(\lambda)\Big\| = O\Big(\frac{1}{n^2}\Big).$$

In Section 5 it is proved that $l(\lambda) := \mathrm{tr}_m(L(\lambda \mathbf{1}_m))$ gives rise to a compactly supported distribution, Λ, such that

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(S_n)\} = (\mathrm{tr}_m \otimes \tau)\phi(s) + \frac{1}{n}\Lambda(\phi) + O\left(\frac{1}{n^2}\right)$$

for any $\phi \in C_c^\infty(\mathbb{R})$ (still with the same notation as in Section 2). In fact, $\mathrm{supp}(\Lambda) \subseteq \sigma(s)$, so $\Lambda(\phi) = 0$ for any $\phi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\phi) \cap \sigma(s) = \emptyset$. It follows that for $\psi \in C^\infty(\mathbb{R})$ such that ψ is constant outside a compact set of $\mathbb{R}$ and $\mathrm{supp}(\psi) \cap \sigma(s) = \emptyset$,

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n)\} = O\left(\frac{1}{n^2}\right)$$

- an estimate similar to the one obtained by Haagerup and Thorbjørnsen in [HT2, Lemma 6.3] and a cornerstone of the proof of Theorem A.

In Section 7 we consider yet two other ensembles which are random matrix ensembles with quaternionic entries. Remember that the quaternions is the division ring, $\mathbb{H}$, which is, as a vector space over the real numbers, spanned by four linearly independent vectors, $\mathbf{1}, j, k, l$, satisfying the identities

$$j^2 = k^2 = l^2 = -\mathbf{1}$$

and

$$jk = -kj = l, \qquad kl = -lk = j, \qquad lj = -jl = k.$$

We denote by $\mathbb{H}^{\mathbb{C}}$ the complexification of $\mathbb{H}$.

It is well known that

$$\mathbf{1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad j \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad k \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad l \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

defines a ring homomorphism which is an embedding of $\mathbb{H}$ into $M_2(\mathbb{C})$.

(vi) $\mathrm{GSE}(n, \sigma^2)$ is the set of random matrices $X \to M_n(\mathbb{H}) = \mathbb{H} \otimes_{\mathbb{R}} M_n(\mathbb{R})$ satisfying that

$$X = \mathbf{1} \otimes V + j \otimes (iW) + k \otimes (iY) + l \otimes (iZ),$$

for some $V \in \mathrm{GOE}(n, \frac{\sigma^2}{4})$ and some $W, Y, Z \in \mathrm{GOE}^*(n, \frac{\sigma^2}{4})$, where V, W, Y and Z are stochastically independent. If we identify $\mathbb{H}$ with a real sub-algebra of $M_2(\mathbb{C})$ in the way described above, then

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes V + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes (iW) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (iY) + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes (iZ),$$

as a random matrix taking values in $M_2(\mathbb{C}) \otimes M_n(\mathbb{C})$, and this shall be our preferred description of $\mathrm{GSE}(n, \sigma^2)$.

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(vii) $\text{GSE}^*(n, \sigma^2)$ is the set of random matrices $X \to M_n(\mathbb{H}^{\mathbb{C}}) = \mathbb{H} \otimes_{\mathbb{R}} M_n(\mathbb{C})$ satisfying that

$$X = \mathbf{1} \otimes V + j \otimes (\mathrm{i}W) + k \otimes (\mathrm{i}Y) + l \otimes (\mathrm{i}Z),$$

for some $V \in \text{GOE}^*(n, \frac{\sigma^2}{4})$ and some $W, Y, Z \in \text{GOE}(n, \frac{\sigma^2}{4})$, where V, W, Y and Z are stochastically independent. Again, we shall preferably consider X as a random matrix taking values in $M_2(\mathbb{C}) \otimes M_n(\mathbb{C})$ with

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes V + \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \otimes (\mathrm{i}W) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (\mathrm{i}Y) + \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \otimes (\mathrm{i}Z).$$

Up to scaling $\text{GSE}(n, \sigma^2)$ is the same as the Gaussian Symplectic Ensemble from [Me].

Whenever $\{X_j | j \in J\}$ is a family of random matrices in $\text{GSE}(n, \sigma^2) \cup \text{GSE}^*(n, \sigma^2)$, we shall say that the X_j's are stochastically independent, if and only if the V_j's, the W_j's, the Y_j's and the Z_j's form a set of stochastically independent random matrices.

Finally, for the sake of completeness we define:

(viii) $\text{GRM}^{\mathbb{H}}(n, \sigma^2)$ is the set of random matrices $Y \to M_n(\mathbb{H})$ satisfying that

$$Y = \mathbf{1} \otimes V + j \otimes W + k \otimes X + l \otimes Z$$

for some stochastically independent random matrices V, W, X and Z from $\text{GRM}^{\mathbb{R}}(n, \frac{\sigma^2}{4})$. Equivalently,

$$Y = \frac{1}{\sqrt{2}}(X_1 + \mathrm{i}X_2), \tag{1.4}$$

for some stochastically independent random matrices $X_1 \in \text{GSE}(n, \sigma^2)$ and $X_2 \in \text{GSE}^*(n, \sigma^2)$.

We apply Theorem A to obtain:

Theorem B. Let $r, s \in \mathbb{N}_0$ with $r + s \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ be stochastically independent random matrices defined on $(\Omega, \mathcal{F}, P)$ such that $X_1^{(n)}, \ldots, X_r^{(n)} \in \text{GSE}(n, \frac{1}{n})$ and $X_{r+1}^{(n)}, \ldots, X_{r+s}^{(n)} \in \text{GSE}^*(n, \frac{1}{n})$. Furthermore, let $(\mathcal{A}, \tau)$ be a C^*-probability space with τ a faithful state on $\mathcal{A}$, and let $\{x_1, \ldots, x_{r+s}\}$ be a semicircular system in $(\mathcal{A}, \tau)$. Then there is a P-null set $N \subseteq \Omega$ such that for any $\omega \in \Omega \setminus N$ and for any polynomial p in $r + s$ non-commuting variables:

$$\lim_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))\| = \|p(x_1, \ldots, x_{r+s})\|.$$

Haagerup and Thorbjørnsen apply [La, Proposition 4.1] and their main Theorem to prove that for $p \in \mathbb{N}$ and $Y_n \in \text{GRM}(n, \frac{1}{n})$,

$$\lim_{n \to \infty} \|Y_n^p\| = \left(\frac{(p+1)^{(p+1)}}{p^p} \right)^{\frac{1}{2}}$$

almost surely (cf. [HT2, Corollary 9.7]). Similarly, with the aid of [La, Proposition 4.1] and the identities (1.1) and (1.4), the following corollary follows from Theorem A and Theorem B:

Corollary. Let $p \in \mathbb{N}$, and for each $n \in \mathbb{N}$, let $Y_n \in \mathrm{GRM}^{\mathbb{R}}(n, \frac{1}{n})$ ($Y_n \in \mathrm{GRM}^{\mathbb{H}}(n, \frac{1}{n})$, respectively). Then, in both cases,

$$\lim_{n \to \infty} \|Y_n^p\| = \left(\frac{(p+1)^{(p+1)}}{p^p} \right)^{\frac{1}{2}}$$

almost surely.

2 Master equation and master inequality in the real case.

Throughout this section let r, $m \in \mathbb{N}$ and $a_0, a_1, \ldots, a_r \in M_m(\mathbb{C})$ with $a_0^* = a_0$, and for each positive integer, n, let $Y_1^{(n)}, \ldots, Y_r^{(n)}$ be independent random matrices from $\mathrm{GRM}^{\mathbb{R}}(n, \frac{1}{n})$. Then define the self-adjoint random matrix S_n taking values in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ by

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{j=1}^{r} (a_j \otimes Y_j^{(n)} + a_j^* \otimes Y_j^{(n)*}). \tag{2.1}$$

Also, for $\lambda \in M_m(\mathbb{C})$ set

$$\mathrm{Im}\lambda = \frac{1}{2\mathrm{i}}(\lambda - \lambda^*),$$

and for $\mathrm{Im}\lambda > 0$ (i.e. $\mathrm{Im}\lambda$ is positive definite) define

$$H_n(\lambda) = (\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]. \tag{2.2}$$

We denote by $e_{kl}^{(m)}$ the matrix unit in $M_m(\mathbb{C})$ with 1 at entry (k, l) and 0 at all other entries.

For an element $u \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ we let u^t denote the transpose of u w.r.t. the natural identification of $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ with $M_{mn}(\mathbb{C})$. Note that for $a \in M_m(\mathbb{C})$ and $b \in M_n(\mathbb{C})$, $(a \otimes b)^t = a^t \otimes b^t$.

Finally, for any invertible square matrix a we denote by a^{-t} the transpose of a^{-1}.

Applying the methods of Haagerup and Thorbjørnsen from [HT2] we prove

2.1 Theorem. (Master equation) For every positive integer n and every $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda > 0$

$$\mathbb{E}\{(a_0 - \lambda)H_n(\lambda) + \sum_{j=1}^{r}[a_j H_n(\lambda)a_j^* H_n(\lambda) + a_j^* H_n(\lambda)a_j H_n(\lambda)] + \mathbf{1}_m\} = -\frac{1}{n}R_n(\lambda), \quad (2.3)$$

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where

$$R_n(\lambda) = \sum_{j=1}^{r} \sum_{k,l=1}^{m} a_j e_{kl}^{(m)} \mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)} a_j \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\}$$

$$+ \sum_{j=1}^{r} \sum_{k,l=1}^{m} a_j^* e_{kl}^{(m)} \mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)} a_j^* \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\}.$$

$$(2.4)$$

Proof. Consider a fixed $n \in \mathbb{N}$. For $1 \le j \le r$ let $\{(Y_j^{(n)})_{kl}\}_{1 \le k,l \le n}$ denote the entries of $Y_j^{(n)}$. By [HT2, Lemma 3.3] and [HT2, Lemma 3.4],

$$\mathbb{E}\{(Y_j^{(n)})_{kl}(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\} = \frac{1}{n}\mathbb{E}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\lambda \otimes \mathbf{1}_n - S_n - t(a_j \otimes e_{kl}^{(n)} + a_j^* \otimes e_{lk}^{(n)}))^{-1}\right\}$$

$$= \frac{1}{n}\mathbb{E}\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{kl}^{(n)} + a_j^* \otimes e_{lk}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\}.$$

Since

$$Y_j^{(n)} = \sum_{k,l=1}^{n} (Y_j^{(n)})_{kl} e_{kl}^{(n)},$$

it follows that

$$\mathbb{E}\{(a_j \otimes Y_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\} =$$

$$\frac{1}{n}\sum_{k,l=1}^{n}(a_j \otimes e_{kl}^{(n)})\mathbb{E}\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{kl}^{(n)} + a_j^* \otimes e_{lk}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\},$$

and

$$\mathbb{E}\{(a_j^* \otimes Y_j^{(n)*})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\} =$$

$$\frac{1}{n}\sum_{k,l=1}^{n}(a_j^* \otimes e_{lk}^{(n)})\mathbb{E}\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{kl}^{(n)} + a_j^* \otimes e_{lk}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\}.$$

As in the proof of [HT2, Lemma 3.5], for every $m \times m$ matrix b,

$$\sum_{k,l=1}^{n}(\mathbf{1}_m \otimes e_{kl}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(b \otimes e_{lk}^{(n)}) = n\, H_n(\lambda) b \otimes \mathbf{1}_n.$$

Also observe that for any elementary tensor $x \otimes y \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$,

$$\sum_{k,l=1}^{n}(\mathbf{1}_m \otimes e_{kl}^{(n)})(x \otimes y)(\mathbf{1}_m \otimes e_{kl}^{(n)}) = x \otimes y^t = (x^t \otimes y)^t$$

$$= \left(\sum_{k,l=1}^{m}(e_{kl}^{(m)} \otimes \mathbf{1}_n)(x \otimes y)(e_{kl}^{(m)} \otimes \mathbf{1}_n)\right)^t$$

$$= \sum_{k,l=1}^{m}(e_{lk}^{(m)} \otimes \mathbf{1}_n)(x \otimes y)^t(e_{lk}^{(m)} \otimes \mathbf{1}_n).$$

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Hence,

$$\sum_{k,l=1}^{n} (\mathbf{1}_m \otimes e_{kl}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(\mathbf{1}_m \otimes e_{kl}^{(n)}) = \sum_{k,l=1}^{m} (e_{kl}^{(m)} \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)} \otimes \mathbf{1}_n).$$

Combining the above observations we find that

$$\mathbb{E}\{(a_j \otimes Y_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\} = \mathbb{E}\{(a_j H_n(\lambda) a_j^* \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\}+$$

$$\frac{1}{n} \sum_{k,l=1}^{m} (a_j e_{kl}^{(m)} \otimes \mathbf{1}_n)\mathbb{E}\{(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)} a_j \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\},$$

and

$$\mathbb{E}\{(a_j^* \otimes Y_j^{(n)^*})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\} = \mathbb{E}\{(a_j^* H_n(\lambda) a_j \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\}+$$

$$\frac{1}{n} \sum_{k,l=1}^{m} (a_j^* e_{kl}^{(m)} \otimes \mathbf{1}_n)\mathbb{E}\{(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)} a_j^* \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\}.$$

Applying now $(\mathrm{id}_m \otimes \mathrm{tr}_n)$ to both sides of the equations above we get

$$\mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[(a_j \otimes Y_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\} = \mathbb{E}\{(a_j H_n(\lambda) a_j^* H_n(\lambda)\}+$$

$$\frac{1}{n} \sum_{k,l=1}^{m} a_j e_{kl}^{(m)}\mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)} a_j \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\},$$

$$\mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[(a_j^* \otimes Y_j^{(n)^*})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\} = \mathbb{E}\{(a_j^* H_n(\lambda) a_j H_n(\lambda)\}+$$

$$\frac{1}{n} \sum_{k,l=1}^{m} (a_j^* e_{kl}^{(m)}\mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)} a_j^* \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\}.$$

Since
$$\mathbb{E}\{(a_0 - \lambda)H_n(\lambda)\} = \mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)((a_0 - \lambda) \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\},$$

we conclude that

$$\mathbb{E}\{(a_0 - \lambda)H_n(\lambda) + \sum_{j=1}^{r} a_j H_n(\lambda) a_j^* H_n(\lambda) + a_j^* H_n(\lambda) a_j H_n(\lambda)\} =$$

$$\mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[((a_0 - \lambda) \otimes \mathbf{1}_n + \sum_{j=1}^{r} a_j \otimes Y_j^{(n)} + a_j^* \otimes Y_j^{(n)^*})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\} - \frac{1}{n}R_n(\lambda).$$

Hence,

$$\mathbb{E}\{(a_0 - \lambda)H_n(\lambda) + \sum_{j=1}^{r}[a_j H_n(\lambda) a_j^* H_n(\lambda) + a_j^* H_n(\lambda) a_j H_n(\lambda)] + \mathbf{1}_m\} = -\frac{1}{n}R_n(\lambda),$$

and the proof is complete. ∎

Now, let $(\mathcal{A}, \tau)$ be a C^*-probability space, and let $y_1, \ldots, y_r$ be a circular system in $(\mathcal{A}, \tau)$, i.e. with

$$
\begin{aligned}
x_j &= \tfrac{1}{\sqrt{2}}(y_j + y_j^*), &\quad (1 \le j \le r) \\
x_{j+r} &= \tfrac{1}{\mathrm{i}\sqrt{2}}(y_j - y_j^*), &\quad (1 \le j \le r)
\end{aligned}
$$

$x_1, \ldots, x_{2r}$ form a semicircular system. Define

$$
s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r} (a_j \otimes y_j + a_j^* \otimes y_j^*). \tag{2.5}
$$

Note that

$$
s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r} \left(\frac{1}{\sqrt{2}}(a_j + a_j^*) \otimes x_j + \frac{1}{\mathrm{i}\sqrt{2}}(a_j - a_j^*) \otimes x_{j+r} \right).
$$

For $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda > 0$ define

$$
G(\lambda) = (\mathrm{id}_m \otimes \tau)[(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]. \tag{2.6}
$$

According to [HT2, Lemma 5.4] we have the following identity

$$
\lambda = a_0 + \sum_{j=1}^{r} \left[\frac{1}{\sqrt{2}}(a_j + a_j^*) G(\lambda) \frac{1}{\sqrt{2}}(a_j + a_j^*) + \frac{1}{\mathrm{i}\sqrt{2}}(a_j - a_j^*) G(\lambda) \frac{\mathrm{i}}{\sqrt{2}}(a_j - a_j^*) \right] + G(\lambda)^{-1}, \tag{2.7}
$$

or equivalently,

$$
\lambda = a_0 + \sum_{j=1}^{r} [a_j G(\lambda) a_j^* + a_j^* G(\lambda) a_j] + G(\lambda)^{-1}. \tag{2.8}
$$

2.2 Remark. Let r and n be positive integers, and define an isomorphism $\psi_0 : M_n(\mathbb{C}) \to \mathbb{C}^{n^2}$ by

$$
\psi_0((a_{kl})_{1 \le k,l \le n}) = ((a_{kl})_{1 \le k,l \le n}).
$$

for $(a_{kl})_{1 \le k,l \le n} \in M_n(\mathbb{C})$. ψ_0 has a natural extension to a linear isomorphism between $M_n(\mathbb{C})^r$ and $\mathbb{C}^{rn^2}$, which we denote by ψ:

$$
\psi(A_1, \ldots, A_r) = (\psi_0(A_1), \ldots, \psi_0(A_r)), \qquad (A_1, \ldots, A_r \in M_n(\mathbb{C})).
$$

Define a norm $\| \cdot \|_e$ on $M_n(\mathbb{C})^r$ by

$$
\|(A_1, \ldots, A_r)\|_e^2 = \mathrm{Tr}_n \left(\sum_{i=1}^{r} A_i^* A_i \right), \qquad (A_1, \ldots, A_r \in M_n(\mathbb{C})),
$$

and note that ψ is an isometry with respect to this norm and the Euclidian norm on $\mathbb{C}^{rn^2}$.

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If $Y_1^{(n)}, \ldots, Y_r^{(n)}$ are independent random matrices from $\mathrm{GRM}^{\mathbb{R}}(n, \frac{1}{n})$, then $\mathbb{Y} = \psi(Y_1^{(n)}, \ldots, Y_r^{(n)})$ is a random variable taking values in $\mathbb{R}^{rn^2}$, and the distribution of $\mathbb{Y}$ on $\mathbb{R}^{rn^2}$ is $\nu \otimes \cdots \otimes \nu$ (rn^2 terms), where ν denotes the Gaussian distribution $N(0, \frac{1}{n})$ on $\mathbb{R}$. By the Gaussian Poincaré inequality (cf. [HT2, Remark 4.3]), if $\tilde{f} : \mathbb{R}^{rn^2} \to \mathbb{C}$ is a C^1-function such that $\tilde{f}$ and $\mathrm{grad}\tilde{f}$ are polynomially bounded, then, with $f = \tilde{f}(\psi) : M_n(\mathbb{C})^r \to \mathbb{C}$:

$$\mathbb{V}\{f(Y_1^{(n)}, \ldots, Y_r^{(n)})\} \leq \frac{1}{n}\mathbb{E}\{\|\mathrm{grad}f(Y_1^{(n)}, \ldots, Y_r^{(n)})\|_e^2\},$$

where $\mathbb{V}\{g\} = \mathbb{E}\{|g - \mathbb{E}\{g\}|^2\}$ for any function $g \in L^2(\Omega, P)$.

2.3 Lemma. *Let r, m and n be positive integers, let $a_1, \ldots, a_r \in M_m(\mathbb{C})$, and let $w_1, \ldots, w_r \in M_n(\mathbb{C})$. Then, with $w = (w_1, \ldots, w_r)$,*

$$\Big\| \sum_{i=1}^r a_i \otimes w_i + a_i^* \otimes w_i^* \Big\|_{2, \mathrm{Tr}_m \otimes \mathrm{Tr}_n} \leq 2m^{1/2} \Big(\sum_{i=1}^r \|a_i\|^2 \Big)^{1/2} \|w\|_e,$$

where $\| \cdot \|_{2, \mathrm{Tr}_m \otimes \mathrm{Tr}_n}$ denotes the Hilbert-Schmidt norm on $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$.

Proof. This is a simple application of the Cauchy-Schwartz inequality for the standard inner product on $\mathbb{C}^r$:

$$
\begin{aligned}
\Big\| \sum_{i=1}^r a_i \otimes w_i + a_i^* \otimes w_i^* \Big\|_{2, \mathrm{Tr}_m \otimes \mathrm{Tr}_n} &\leq 2 \sum_{i=1}^r \|a_i\|_{2, \mathrm{Tr}_m} \|w_i\|_{2, \mathrm{Tr}_n} \\
&\leq 2 \Big(\sum_{i=1}^r \|a_i\|_{2, \mathrm{Tr}_m}^2 \Big)^{1/2} \Big(\sum_{i=1}^r \|w_i\|_{2, \mathrm{Tr}_n}^2 \Big)^{1/2} \\
&= 2 \Big(\sum_{i=1}^r \mathrm{Tr}_m(a_i^* a_i) \Big)^{1/2} \|w\|_e \\
&\leq 2m^{1/2} \Big\| \sum_{i=1}^r a_i^* a_i \Big\|^{1/2} \|w\|_e \\
&\leq 2m^{1/2} \Big(\sum_{i=1}^r \|a_i\|^2 \Big)^{1/2} \|w\|_e. \qquad \blacksquare
\end{aligned}
$$

2.4 Theorem. (Master inequality) There is a constant $C_1 < \infty$ such that for every positive integer n and for every $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda > 0$,

$$\Big\| a_0 + \sum_{j=1}^r [a_j G_n(\lambda) a_j^* + a_j^* G_n(\lambda) a_j] + G_n(\lambda)^{-1} - \lambda + \frac{1}{n} R_n(\lambda) G_n(\lambda)^{-1} \Big\|$$

$$\leq \frac{C_1}{n^2} (\|\lambda\| + K)^2 \|(\mathrm{Im}\lambda)^{-1}\|^5,$$

(2.9)

where $K = \|a_0\| + 16 \sum_{j=1}^r \|a_j\|$.

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Proof. Let $n \in \mathbb{N}$, and let $\lambda \in M_m(\mathbb{C})$ with $\operatorname{Im}\lambda > 0$. Define

$$K_n(\lambda) = H_n(\lambda) - G_n(\lambda). \tag{2.10}$$

According to Theorem 2.1

$$\mathbb{E}\Big\{ \sum_{j=1}^{r} a_j K_n(\lambda) a_j^* K_n(\lambda) + a_j^* K_n(\lambda) a_j K_n(\lambda) \Big\} =$$

$$-\Big(\sum_{j=1}^{r} [a_j G_n(\lambda) a_j^* G_n(\lambda) + a_j^* G_n(\lambda) a_j G_n(\lambda)] + (a_0 - \lambda) G_n(\lambda) + \mathbf{1}_m \Big) - \frac{1}{n} R_n(\lambda),$$

i.e.

$$-\mathbb{E}\Big\{ \sum_{j=1}^{r} a_j K_n(\lambda) a_j^* K_n(\lambda) + a_j^* K_n(\lambda) a_j K_n(\lambda) \Big\} G_n(\lambda)^{-1} =$$

$$a_0 + \sum_{j=1}^{r} [a_j G_n(\lambda) a_j^* + a_j^* G_n(\lambda) a_j] + G_n(\lambda)^{-1} - \lambda + \frac{1}{n} R_n(\lambda) G_n(\lambda)^{-1},$$

which implies that

$$\Big\| a_0 + \sum_{j=1}^{r} [a_j G_n(\lambda) a_j^* + a_j^* G_n(\lambda) a_j] + G_n(\lambda)^{-1} - \lambda + \frac{1}{n} R_n(\lambda) G_n(\lambda)^{-1} \Big\|$$

$$\leq \mathbb{E}\Big\{ \Big\| \sum_{j=1}^{r} a_j K_n(\lambda) a_j^* + a_j^* K_n(\lambda) a_j \Big\| \|K_n(\lambda)\| \Big\} \|G_n(\lambda)^{-1}\|.$$

The mappings $v \mapsto a_j v a_j^*$, $v \mapsto a_j^* v a_j$, $1 \leq j \leq r$, are all completely positive. Hence, the sum of these mappings is also completely positive, so it attains it norm at the unit of $M_m(\mathbb{C})$, and we have

$$\Big\| a_0 + \sum_{j=1}^{r} [a_j G_n(\lambda) a_j^* + a_j^* G_n(\lambda) a_j] + G_n(\lambda)^{-1} - \lambda + \frac{1}{n} R_n(\lambda) G_n(\lambda)^{-1} \Big\|$$

$$\leq \Big\| \sum_{j=1}^{r} a_j a_j^* + a_j^* a_j \Big\| \mathbb{E}\{\|K_n(\lambda)\|^2\} \|G_n(\lambda)^{-1}\|$$

$$\leq 2 \sum_{j=1}^{r} \|a_j\|^2 \mathbb{E}\{\|K_n(\lambda)\|_{2,\operatorname{Tr}_m}^2\} \|G_n(\lambda)^{-1}\|,$$

where $\|K_n(\lambda)\|_{2,\operatorname{Tr}_m}$ denotes the Hilbert-Schmidt norm of $K_n(\lambda)$.

Now, for $Y \in \operatorname{GRM}^{\mathbb{R}}(n, \frac{1}{n})$ we can choose $Z \in \operatorname{GRM}(n, \frac{1}{n})$ such that

$$Y = \frac{1}{\sqrt{2}}(Z + \overline{Z}),$$

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where $\overline{Z} = (Z^*)^t$. The random matrix Z may be expressed in terms of two indenpendent random matrices $X_1, X_2 \in \mathrm{SGRM}(n, \frac{1}{n})$:

$$Z = \frac{1}{\sqrt{2}}(X_1 + \mathrm{i}X_2).$$

It follows from [HT2, Lemma 5.1] that

$$\mathbb{E}\{\|Y\|\} \leq 2\mathbb{E}\{\|X_1\|\} \leq 8.$$

By arguments similar to those presented in [HT2, Proof of Proposition 5.2],

$$\|G_n(\lambda)^{-1}\| \leq (\|\lambda\| + K)^2 \|(\mathrm{Im}\lambda)^{-1}\|, \tag{2.11}$$

where $K = \|a_0\| + 16\sum_{j=1}^{r} \|a_j\|$.

To get an estimate of $\mathbb{E}\{\|K_n(\lambda)\|_{2,\mathrm{Tr}_m}^2\}$ we follow the lines of [HT2, Proof of Theorem 4.5] and obtain:

$$\mathbb{E}\{\|K_n(\lambda)\|_{2,\mathrm{Tr}_m}^2\} \leq \frac{1}{n}\sum_{j,k=1}^{m} \mathbb{E}\{\|(\mathrm{grad}f_{n,j,k})(Y_1^{(n)}, \ldots, Y_r^{(n)})\|_e^2\}, \tag{2.12}$$

where each $f_{n,j,k} : M_n(\mathbb{C})^r \to \mathbb{C}$ is defined by

$$f_{n,j,k}(v_1, \ldots, v_r) = m(\mathrm{tr}_m \otimes \mathrm{tr}_n)[(e_{jk}^{(m)} \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - a_0 \otimes \mathbf{1}_n - \sum_{i=1}^{r} a_i \otimes v_i + a_i^* \otimes v_i^*)^{-1}].$$

Let $1 \leq j, k \leq m$, let $v = (v_1, \ldots, v_r) \in M_n(\mathbb{C})^r$, and let $w = (w_1, \ldots, w_r) \in M_n(\mathbb{C})^r$ with $\|w\|_e = 1$. Proceeding as in [HT2, Proof of Theorem 4.5] we find that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,j,k}(v + tw)\right|^2 \leq \frac{1}{n}\Big\|\sum_{i=1}^{r} a_i \otimes w_i + a_i^* \otimes w_i^*\Big\|_{2,\mathrm{Tr}_m \otimes \mathrm{Tr}_n}^2 \|(\mathrm{Im}\lambda)^{-1}\|^4.$$

Hence, by Lemma 2.3

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,j,k}(v + tw)\right|^2 \leq \frac{4m}{n}\sum_{i=1}^{r} \|a_i\|^2 \|(\mathrm{Im}\lambda)^{-1}\|^4, \tag{2.13}$$

and since w was arbitrary,

$$\|\mathrm{grad}f_{n,j,k}(v)\|_e^2 \leq \frac{4m}{n}\sum_{i=1}^{r} \|a_i\|^2 \|(\mathrm{Im}\lambda)^{-1}\|^4. \tag{2.14}$$

Inserting this into (2.12) we obtain

$$\mathbb{E}\{\|K_n(\lambda)\|_{2,\mathrm{Tr}_m}^2\} \leq \frac{4m^3}{n^2}\sum_{i=1}^{r} \|a_i\|^2 \|(\mathrm{Im}\lambda)^{-1}\|^4. \tag{2.15}$$

Finally,

$$\left\| a_0 + \sum_{j=1}^{r}[a_j G_n(\lambda)a_j^* + a_j^* G_n(\lambda)a_j] + G_n(\lambda)^{-1} - \lambda + \frac{1}{n}R_n(\lambda)G_n(\lambda)^{-1} \right\|$$

$$\leq 2\sum_{j=1}^{r}\|a_j\|^2 \frac{4m^3}{n^2} \sum_{i=1}^{r}\|a_i\|^2 \|(\mathrm{Im}\lambda)^{-1}\|^4 (\|\lambda\| + K)^2 \|(\mathrm{Im}\lambda)^{-1}\|$$

$$= \frac{8m^3}{n^2}\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)^2 (\|\lambda\| + K)^2 \|(\mathrm{Im}\lambda)^{-1}\|^5,$$

from which the theorem follows.   ∎

2.5 Corollary. *There is a constant $C_1' < \infty$ such that for every positive integer n and for every $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda > 0$,*

$$\left\| a_0 + \sum_{j=1}^{r}[a_j G_n(\lambda)a_j^* + a_j^* G_n(\lambda)a_j] + G_n(\lambda)^{-1} - \lambda \right\|$$

(2.16)

$$\leq \frac{C_1'}{n}(\|\lambda\| + K)^2 (\|(\mathit{Im}\lambda)^{-1}\|^5 + \|(\mathit{Im}\lambda)^{-1}\|^3),$$

where $K = \|a_0\| + 16\sum_{j=1}^{r}\|a_j\|$.

Proof. By application of [HT2, Lemma 3.1] we find that

$$\|R_n(\lambda)\| \leq 2\sum_{k,l=1}^{m}\sum_{j=1}^{r}\|a_j\|\mathbb{E}\{\|(\lambda \otimes \mathbf{1}_n - S_n)^{-t}\|\|a_j\|\|(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\|\}$$

$$\leq 2m^2\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)\|(\mathrm{Im}\lambda)^{-1}\|^2.$$

Then, by Theorem 2.4 and by (2.11)

$$\left\| a_0 + \sum_{j=1}^{r}[a_j G_n(\lambda)a_j^* + a_j^* G_n(\lambda)a_j] + G_n(\lambda)^{-1} - \lambda \right\|$$

$$\leq \frac{C_1}{n^2}(\|\lambda\| + K)^2 \|(\mathrm{Im}\lambda)^{-1}\|^5 + \frac{1}{n}\|R_n(\lambda)\|\|G_n(\lambda)^{-1}\|$$

$$\leq \frac{C_1}{n}(\|\lambda\| + K)^2 \|(\mathrm{Im}\lambda)^{-1}\|^5 + \frac{2m^2}{n}\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)\|(\mathrm{Im}\lambda)^{-1}\|^2 (\|\lambda\| + K)^2 \|(\mathrm{Im}\lambda)^{-1}\|,$$

and the corollary follows.   ∎

3   Estimation of $\|G_n(\lambda) - G(\lambda)\|$.

We stick to the notation introduced in the previous section and define the subset $\mathcal{O}$ of $M_m(\mathbb{C})$ by

$$\mathcal{O} = \{\lambda \in M_m(\mathbb{C}) | \mathrm{Im}\lambda > 0\}, \tag{3.1}$$

and for $\lambda \in \mathcal{O}$ put

$$\Lambda_n(\lambda) \;=\; a_0 + \sum_{j=1}^{r}[a_j G_n(\lambda)a_j^* + a_j^* G_n(\lambda)a_j] + G_n(\lambda)^{-1}, \tag{3.2}$$

$$\varepsilon(\lambda) \;=\; \|(\mathrm{Im}\lambda)^{-1}\|^{-1}. \tag{3.3}$$

Finally let

$$\mathcal{O}_n' \;=\; \left\{\lambda \in \mathcal{O}\,\Big|\,\frac{C_1'}{n}(K + \|\lambda\|)^2(\varepsilon(\lambda)^{-6} + \varepsilon(\lambda)^{-4}) < \frac{1}{2}\right\}. \tag{3.4}$$

By application of Corollary 2.5 and the methods of [HT2, Proof of Lemma 5.5] one finds that for any $\lambda \in \mathcal{O}_n'$,

$$\mathrm{Im}\Lambda_n(\lambda) \geq \frac{\varepsilon(\lambda)}{2}\mathbf{1}_m.$$

Thus, $\Lambda_n(\lambda) \in \mathcal{O}$ with

$$\|(\mathrm{Im}\Lambda_n(\lambda))^{-1}\| \leq 2\|(\mathrm{Im}\lambda)^{-1}\|, \tag{3.5}$$

and by (2.8),

$$a_0 + \sum_{j=1}^{r}[a_j G_n(\lambda)a_j^* + a_j^* G_n(\lambda)a_j] + G_n(\lambda)^{-1} =$$

$$a_0 + \sum_{j=1}^{r}[a_j G(\Lambda_n(\lambda))a_j^* + a_j^* G(\Lambda_n(\lambda))a_j] + G(\Lambda_n(\lambda))^{-1}.$$

3.1 Proposition. *Let $n \in \mathbb{N}$. Then for all $\lambda \in \mathcal{O}_n'$,*

$$G(\Lambda_n(\lambda)) = G_n(\lambda).$$

Proof. The proof is almost identical to [HT2, Proof of Proposition 5.5]. Only a few modifications are necessary to make it work in this case too.

Making use of the fact that the function $t \mapsto (K+t)^2(t^{-6}+t^{-4})$, $t > 0$, is continuous and strictly decreasing, it follows as in [HT2] that

  (a) $\mathcal{O}_n'$ is an open connected subset of $M_m(\mathbb{C})$.

With

$$\mathcal{O}_n'' = \{\lambda \in \mathcal{O}_n' | \varepsilon(\lambda) > 2(\sum_{j=1}^{r}\|a_j\|^2)^{\frac{1}{2}}\},$$

which is an open, nonempty subset of $\mathcal{O}_n'$, one gets, as in [HT2], that

  (b) $G(\Lambda_n(\lambda)) = G_n(\lambda)$ for all $\lambda \in \mathcal{O}_n''$.

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Finally, apply the principle of uniqueness of analytic continuation. ■

Taking Corollary 2.5 into account and proceeding as in [HT2, Proof of Theorem 5.7] one gets:

3.2 Theorem. *There is a constant $C_2 < \infty$ such that for any $\lambda \in \mathcal{O}$ and for any positive integer n*

$$\|G_n(\lambda) - G(\lambda)\| \leq \frac{C_2}{n}(\|\lambda\| + K)^2(\|(\mathrm{Im}\lambda)^{-1}\|^7 + \|(\mathrm{Im}\lambda)^{-1}\|^5), \tag{3.6}$$

where $K = \|a_0\| + 16\sum_{j=1}^r \|a_j\|$.

Before stating the next corollary we introduce some notation. Let $\mathcal{U}$ be an open subset of $M_m(\mathbb{C})$, and let $\phi : \mathcal{U} \to M_m(\mathbb{C})$ be a complex differentiable map. For $v \in \mathcal{U}$ we denote by $\phi'(v)$ the differential of ϕ at v, i.e. $\phi'(v) : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is the (unique) linear map satisfying that for every differentiable curve α defined in a neighbourhood of zero, $\alpha : (-\varepsilon, \varepsilon) \to \mathcal{U}$, with $\alpha(0) = v$, the tangent vector $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\phi(\alpha(t))$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi(\alpha(t)) = \phi'(v)[\alpha'(0)].$$

3.3 Corollary. *Let $\lambda \in \mathcal{O}$, and let C_2 be as in Theorem 3.2. Then for every positive integer n we have*

$$\|G_n'(\lambda) - G'(\lambda)\| \leq \frac{576C_2}{n}(\|\lambda\| + K)^2(\|(\mathrm{Im}\lambda)^{-1}\|^8 + \|(\mathrm{Im}\lambda)^{-1}\|^6). \tag{3.7}$$

Proof. Let $n \in \mathbb{N}$ and let $x \in M_m(\mathbb{C})$ with $\|x\| = 1$. Note that for every complex number z

$$\mathrm{Im}(\lambda + zx) = \mathrm{Im}\lambda + \mathrm{Im}(zx) \geq \mathrm{Im}\lambda - |z|\mathbf{1}_m.$$

So $\lambda + zx \in \mathcal{O}$ if $|z| < \varepsilon(\lambda) = \|(\mathrm{Im}\lambda)^{-1}\|^{-1}$. Hence, the map

$$z \mapsto G_n(\lambda + zx) - G(\lambda + zx)$$

is well-defined (and analytic) in the open disc in $\mathbb{C}$ of radius $\|(\mathrm{Im}\lambda)^{-1}\|^{-1}$ centered at zero. Put $r = \frac{1}{2}\|(\mathrm{Im}\lambda)^{-1}\|^{-1}$, and define a path γ by $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$. Then, according to the Cauchy formulas for vector valued analytic functions,

$\|(G_n'(\lambda) - G'(\lambda))[x]\|$

$$= \left\|\frac{\mathrm{d}}{\mathrm{d}z}\Big|_{z=0}(G_n(\lambda + zx) - G(\lambda + zx))\right\|$$

$$= \left\|\frac{1}{2\pi i}\int_\gamma \frac{G_n(\lambda + \zeta x) - G(\lambda + \zeta x)}{\zeta^2}\mathrm{d}\zeta\right\|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{\|G_n(\lambda + \gamma(t)x) - G(\lambda + \gamma(t)x)\|}{r}\mathrm{d}t$$

$$\leq \frac{1}{r}\max_{0 \leq t \leq 2\pi}\{\|G_n(\lambda + \gamma(t)x) - G(\lambda + \gamma(t)x)\|\}$$

$$\leq \frac{C_2}{nr}\max_{0 \leq t \leq 2\pi}\{(\|\lambda + \gamma(t)x\| + K)^2 \cdot (\|(\mathrm{Im}(\lambda + \gamma(t)x))^{-1}\|^7 + \|(\mathrm{Im}(\lambda + \gamma(t)x))^{-1}\|^5)\}.$$

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Now, for $t \in [0, 2\pi]$

$$\text{Im}(\lambda + \gamma(t)x) = \text{Im}\lambda + r\text{Im}(e^{it}x) \geq (\varepsilon(\lambda) - r)\mathbf{1}_m = \frac{1}{2}\|(\text{Im}\lambda)^{-1}\|^{-1}\mathbf{1}_m.$$

Hence,

$$\|(\text{Im}(\lambda + \gamma(t)x))^{-1}\| \leq 2\|(\text{Im}\lambda)^{-1}\|.$$

Also, since

$$\|\lambda\|\|(\text{Im}\lambda)^{-1}\| \geq \|\text{Im}\lambda\|\|(\text{Im}\lambda)^{-1}\| \geq 1,$$

we have

$$\|\lambda + \gamma(t)x\| \leq \|\lambda\| + r = \|\lambda\| + \frac{1}{2\|(\text{Im}\lambda)^{-1}\|} \leq \frac{3}{2}\|\lambda\|.$$

It follows that

$$
\begin{aligned}
\|(G'_n(\lambda) - G'(\lambda))[x]\| &\leq \frac{C_2}{n}\left(\frac{3}{2}\|\lambda\| + K\right)^2 (2^8\|(\text{Im}\lambda)^{-1}\|^8 + 2^6\|(\text{Im}\lambda)^{-1}\|^6) \\
&\leq \frac{256C_2}{n}\frac{9}{4}(\|\lambda\| + K)^2(\|(\text{Im}\lambda)^{-1}\|^8 + \|(\text{Im}\lambda)^{-1}\|^6) \\
&= \frac{576C_2}{n}(\|\lambda\| + K)^2(\|(\text{Im}\lambda)^{-1}\|^8 + \|(\text{Im}\lambda)^{-1}\|^6). \qquad \blacksquare
\end{aligned}
$$

4 Estimation of $\|G(\lambda) - G_n(\lambda) + \frac{1}{n}L(\lambda)\|$.

For $a_0, a_1, \ldots, a_r$ in $M_m(\mathbb{C})$ with $a_0 = a_0^*$ (as in the previous sections) define

$$\tilde{a}_j = \begin{pmatrix} \overline{a_j} & 0 \\ 0 & a_j \end{pmatrix} \in M_{2m}(\mathbb{C}), \qquad (0 \leq j \leq r),$$

where $\overline{a_j}$ is the matrix obtained by conjugation of the entries of a_j. Note that $\overline{a_j} = (a_j^*)^t$.

For $\lambda \in M_m(\mathbb{C})$ with $\text{Im}\lambda > 0$ set

$$\tilde{\lambda} = \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda \end{pmatrix} \in M_{2m}(\mathbb{C}),$$

and for $Y_1^{(n)}, \ldots, Y_r^{(n)}$ stochastically independent random matrices from the class $\text{GRM}^{\mathbb{R}}(n, \frac{1}{n})$ define

$$\tilde{S}_n = \tilde{a}_0 \otimes \mathbf{1}_n + \sum_{j=1}^r (\tilde{a}_j \otimes Y_j^{(n)} + \tilde{a}_j^* \otimes Y_j^{(n)*}), \qquad (4.1)$$

$$\tilde{s} = \tilde{a}_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^r (\tilde{a}_j \otimes y_j + \tilde{a}_j^* \otimes y_j^*), \qquad (4.2)$$

$$\tilde{G}_n(\tilde{\lambda}) = \mathbb{E}\{(\text{id}_{2m} \otimes \text{tr}_n)[(\tilde{\lambda} \otimes \mathbf{1}_n - \tilde{S}_n)^{-1}]\}, \qquad (4.3)$$

$$\tilde{G}(\tilde{\lambda}) = (\text{id}_{2m} \otimes \tau)[(\tilde{\lambda} \otimes \mathbf{1}_{\mathcal{A}} - \tilde{s})^{-1}]. \qquad (4.4)$$

Note that $\|\tilde{a}_j\| = \|a_j\|$, $0 \leq j \leq r$, $\|\tilde{\lambda}\| = \|\lambda\|$, and that $\text{Im}\tilde{\lambda} > 0$ with $\|(\text{Im}\tilde{\lambda})^{-1}\| = \|(\text{Im}\lambda)^{-1}\|$. Thus, it follows from the results obtained this far that there is a constant, $\tilde{C}_2 < \infty$, such that for every positive integer n and every $\lambda \in M_m(\mathbb{C})$ with $\text{Im}\lambda > 0$,

$$\|\tilde{G}_n(\tilde{\lambda}) - \tilde{G}(\tilde{\lambda})\| \leq \frac{\tilde{C}_2}{n}(\|\lambda\| + K)^2(\|(\text{Im}\lambda)^{-1}\|^7 + \|(\text{Im}\lambda)^{-1}\|^5), \tag{4.5}$$

where $K = \|a_0\| + 16\sum_{j=1}^{r}\|a_j\|$. The proof of Corollary 3.3 also carries over, so

$$\|\tilde{G}_n'(\tilde{\lambda}) - \tilde{G}'(\tilde{\lambda})\| \leq \frac{576\tilde{C}_2}{n}(\|\lambda\| + K)^2(\|(\text{Im}\lambda)^{-1}\|^8 + \|(\text{Im}\lambda)^{-1}\|^6). \tag{4.6}$$

4.1 Lemma. *For every positive integer n and every $\lambda \in M_m(\mathbb{C})$ with $\text{Im}\lambda > 0$,*

$$R_n(\lambda) = -\sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j e_{kl}^{(m)}(\text{Tr}_2 \otimes \text{id}_m)\left\{\tilde{G}_n'(\tilde{\lambda})\begin{bmatrix}0 & e_{kl}^{(m)}a_j \\ 0 & 0\end{bmatrix}\begin{pmatrix}0 & 0 \\ \mathbf{1}_m & 0\end{pmatrix}\right\}$$

$$-\sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j^* e_{kl}^{(m)}(\text{Tr}_2 \otimes \text{id}_m)\left\{\tilde{G}_n'(\tilde{\lambda})\begin{bmatrix}0 & e_{kl}^{(m)}a_j^* \\ 0 & 0\end{bmatrix}\begin{pmatrix}0 & 0 \\ \mathbf{1}_m & 0\end{pmatrix}\right\}.$$

Proof. By definition

$$R_n(\lambda) = \sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j e_{kl}^{(m)}\mathbb{E}\{(\text{id}_m \otimes \text{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)}a_j \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\}$$

$$+ \sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j^* e_{kl}^{(m)}\mathbb{E}\{(\text{id}_m \otimes \text{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{kl}^{(m)}a_j^* \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\}.$$

Note that $(\lambda \otimes \mathbf{1}_n - S_n)^{-t} = (\lambda^t \otimes \mathbf{1}_n - S_n^t)^{-1}$. Moreover, since the random matrices $Y_j^{(n)}$ have real entries,

$$S_n^t = a_0^t \otimes \mathbf{1}_n + \sum_{j=1}^{r}\left(a_j^t \otimes Y_j^{(n)*} + (a_j^*)^t \otimes Y_j^{(n)}\right).$$

As $a_0 = a_0^*$, it follows that

$$S_n^t = \overline{a_0} \otimes \mathbf{1}_n + \sum_{j=1}^{r}\left(\overline{a_j} \otimes Y_j^{(n)} + \overline{a_j}^t \otimes Y_j^{(n)*}\right).$$

Hence,

$$\tilde{S}_n = \begin{pmatrix}S_n^t & 0 \\ 0 & S_n\end{pmatrix}.$$

Now, standard matrix manipulations reveal that for every $x \in M_m(\mathbb{C})$,

$$\begin{pmatrix}0 & (\lambda \otimes \mathbf{1}_n - S_n)^{-t}(x \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1} - S_n)^{-1} \\ 0 & 0\end{pmatrix} = (\tilde{\lambda} \otimes \mathbf{1}_n - \tilde{S}_n)^{-1}(\tilde{x} \otimes \mathbf{1}_n)(\tilde{\lambda} \otimes \mathbf{1}_n - \tilde{S}_n)^{-1},$$

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where
$$\tilde{x} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_{2m}(\mathbb{C}).$$

Thus, by [HT2, Lemma 3.2],
$$\begin{pmatrix} 0 & (\lambda \otimes \mathbf{1}_n - S_n)^{-t}(x \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1} - S_n)^{-1} \\ 0 & 0 \end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}((\tilde{\lambda} - t\tilde{x}) \otimes \mathbf{1}_n - \tilde{S}_n)^{-1}.$$

It follows that
$$(\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda^t \otimes \mathbf{1}_n - S_n^t)^{-1}(x \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1} - S_n)^{-1}]$$
$$= (\mathrm{Tr}_2 \otimes \mathrm{id}_m)\left\{ \begin{pmatrix} 0 & (\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(x \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1} - S_n)^{-1}] \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}$$
$$= (\mathrm{Tr}_2 \otimes \mathrm{id}_m)\left\{ (\mathrm{id}_{2m} \otimes \mathrm{tr}_n)[(\tilde{\lambda} \otimes \mathbf{1}_n - \tilde{S}_n)^{-1}(x \otimes \mathbf{1}_n)(\tilde{\lambda} \otimes \mathbf{1}_n - \tilde{S}_n)^{-1}] \begin{pmatrix} 0 & 0 \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}$$
$$= (\mathrm{Tr}_2 \otimes \mathrm{id}_m)\left\{ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{id}_{2m} \otimes \mathrm{tr}_n)[((\tilde{\lambda} - t\tilde{x}) \otimes \mathbf{1}_n - \tilde{S}_n)^{-1}] \begin{pmatrix} 0 & 0 \\ \mathbf{1}_m & 0 \end{pmatrix} \right\},$$

and
$$\mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-t}(x \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1} - S_n)^{-1}]\}$$
$$= (\mathrm{Tr}_2 \otimes \mathrm{id}_m)\left\{ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{G}_n(\tilde{\lambda} - t\tilde{x}) \begin{pmatrix} 0 & 0 \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}$$
$$= -(\mathrm{Tr}_2 \otimes \mathrm{id}_m)\left\{ \tilde{G}_n'(\tilde{\lambda})[\tilde{x}] \begin{pmatrix} 0 & 0 \\ \mathbf{1}_m & 0 \end{pmatrix} \right\},$$

and the proof is complete. ∎

4.2 Remark. Inspired by Lemma 4.1 we define $R(\lambda)$, a 'semicircular analogue' of $R_n(\lambda)$:
$$R(\lambda) = -\sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j e_{kl}^{(m)} (\mathrm{Tr}_2 \otimes \mathrm{id}_m)\left\{ \tilde{G}'(\tilde{\lambda})\left[\begin{pmatrix} 0 & e_{kl}^{(m)} a_j \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 0 & 0 \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}$$
$$- \sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j^* e_{kl}^{(m)} (\mathrm{Tr}_2 \otimes \mathrm{id}_m)\left\{ \tilde{G}'(\tilde{\lambda})\left[\begin{pmatrix} 0 & e_{kl}^{(m)} a_j^* \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 0 & 0 \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}.$$

Then, according to (4.6),
$$\|R_n(\lambda) - R(\lambda)\| \leq 2m^2 \sum_{j=1}^{r} \|a_j\| \|(\mathrm{Tr}_2 \otimes \mathrm{id}_m)\| \|\tilde{G}_n'(\tilde{\lambda}) - \tilde{G}'(\tilde{\lambda})\| \|a_j\|$$
$$\leq \frac{2304\tilde{C}_2 m^2}{n}\Big(\sum_{j=1}^{r} \|a_j\|^2\Big)(\|\lambda\| + K)^2(\|(\mathrm{Im}\lambda)^{-1}\|^8 + \|(\mathrm{Im}\lambda)^{-1}\|^6).$$

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And again, matrix manipulations reveal that

$$R(\lambda) = \sum_{j=1}^{r} \sum_{k,l=1}^{m} a_j e_{kl}^{(m)} (\mathrm{id}_m \otimes \tau)[(\lambda^t \otimes \mathbf{1}_{\mathcal{A}} - \overline{s})^{-1} (e_{kl}^{(m)} a_j \otimes \mathbf{1}_{\mathcal{A}})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]$$

$$+ \sum_{j=1}^{r} \sum_{k,l=1}^{m} a_j^* e_{kl}^{(m)} (\mathrm{id}_m \otimes \tau)[(\lambda^t \otimes \mathbf{1}_{\mathcal{A}} - \overline{s})^{-1} (e_{kl}^{(m)} a_j^* \otimes \mathbf{1}_{\mathcal{A}})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}],$$

where

$$\overline{s} = \overline{a_0} \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r} (\overline{a_j} \otimes y_j + \overline{a_j^*} \otimes y_j^*).$$

In particular,

$$\|R(\lambda)\| \leq 2m^2 \Big(\sum_{j=1}^{r} \|a_j\|^2 \Big) \|(\mathrm{Im}\lambda)^{-1}\|^2. \tag{4.7}$$

The same argument applied to (2.4) gives

$$\|R_n(\lambda)\| \leq 2m^2 \Big(\sum_{j=1}^{r} \|a_j\|^2 \Big) \|(\mathrm{Im}\lambda)^{-1}\|^2. \tag{4.8}$$

4.3 Remark. Before proceeding any further we mention that all of the constants introduced this far, i.e. C_1, C_1', C_2 and $\tilde{C}_2$, may be expressed in terms of $m, \|a_0\|, \ldots, \|a_{r-1}\|$ and $\|a_r\|$.

4.4 Theorem. *Let r and m be positive integers, let $a_0, a_1, \ldots, a_r$ be matrices in $M_m(\mathbb{C})$ with $a_0 = a_0^*$, and for each positive integer n let $Y_1^{(n)}, \ldots, Y_r^{(n)}$ be stochastically independent random matrices from $GRM^{\mathbb{R}}(n, \frac{1}{n})$. Furthermore, let $(\mathcal{A}, \tau)$ be a C^*-probability space, and let $y_1, \ldots, y_r$ be a circular system in $(\mathcal{A}, \tau)$. Define*

$$s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r} (a_j \otimes y_j + a_j^* \otimes y_j^*),$$

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{j=1}^{r} (a_j \otimes Y_j^{(n)} + a_j^* \otimes (Y_j^{(n)})^*),$$

and for $\lambda \in \mathcal{O} = \{\lambda \in M_m(\mathbb{C}) | \mathrm{Im}\lambda > 0\}$ *put*

$$
\begin{align}
G_n(\lambda) &= \mathbb{E}\{(id_m \otimes tr_n)[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}]\}, \tag{4.9}\\
G(\lambda) &= (id_m \otimes \tau)[(\lambda \otimes \mathbf{1} - s)^{-1}], \tag{4.10}\\
\overline{s} &= \overline{a_0} \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r}(\overline{a_j} \otimes y_j + \overline{a_j}^* \otimes y_j^*), \tag{4.11}
\end{align}
$$

$$
\begin{align}
R(\lambda) &= \sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j e_{kl}^{(m)}(id_m \otimes \tau)[(\lambda^t \otimes \mathbf{1}_{\mathcal{A}} - \overline{s})^{-1}(e_{kl}^{(m)}a_j \otimes \mathbf{1}_{\mathcal{A}})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}] \notag\\
&\tag{4.12}\\
&+ \sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j^* e_{kl}^{(m)}(id_m \otimes \tau)[(\lambda^t \otimes \mathbf{1}_{\mathcal{A}} - \overline{s})^{-1}(e_{kl}^{(m)}a_j^* \otimes \mathbf{1}_{\mathcal{A}})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}], \notag
\end{align}
$$

and

$$
L(\lambda) = (id_m \otimes \tau)[(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}(R(\lambda)G(\lambda)^{-1} \otimes \mathbf{1})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]. \tag{4.13}
$$

Then there is a polynomial P of degree 13 with non-negative coefficients depending only on $m, \|a_0\|, \ldots, \|a_{r-1}\|$ and $\|a_r\|$, such that for any $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda > 0$ and for any $n \in \mathbb{N}$

$$
\left\|G(\lambda) - G_n(\lambda) + \frac{1}{n}L(\lambda)\right\| \leq \frac{1}{n^2}(\|\lambda\| + K')^8 P(\|(\mathrm{Im}\lambda)^{-1}\|), \tag{4.14}
$$

where $K' = \|a_0\| + 16\sum_{j=1}^{r}\|a_j\| + 1 = K + 1$.

Proof. Consider a fixed $n \in \mathbb{N}$, and at first consider an arbitrary $\lambda \in \mathcal{O}_n'$. With $\Lambda_n(\lambda)$ as previously defined,

$$
\begin{align}
\left\|\Lambda_n(\lambda) - \lambda + \frac{1}{n}R(\lambda)G(\lambda)^{-1}\right\| &\leq \left\|\Lambda_n(\lambda) - \lambda + \frac{1}{n}R_n(\lambda)G_n(\lambda)^{-1}\right\| + \notag\\
&\quad \frac{1}{n}\left\|R_n(\lambda)G_n(\lambda)^{-1} - R(\lambda)G(\lambda)^{-1}\right\| \notag\\
&\leq \left\|\Lambda_n(\lambda) - \lambda + \frac{1}{n}R_n(\lambda)G_n(\lambda)^{-1}\right\| + \notag\\
&\quad \frac{1}{n}\left\|R_n(\lambda)(G_n(\lambda)^{-1} - G(\lambda)^{-1})\right\| + \notag\\
&\quad \frac{1}{n}\left\|(R_n(\lambda) - R(\lambda))G(\lambda)^{-1}\right\|. \notag
\end{align}
$$

Throughout the proof we shall make use of the estimates

$$
\|(\lambda \otimes \mathbf{1} - s)^{-1}\| \leq \|(\mathrm{Im}\lambda)^{-1}\|, \tag{4.15}
$$

$$
\|(\Lambda_n(\lambda) \otimes \mathbf{1} - s)^{-1}\| \leq 2\|(\mathrm{Im}\lambda)^{-1}\|, \tag{4.16}
$$

$$
\|G(\lambda)^{-1}\| \leq (\|\lambda\| + K)^2\|(\mathrm{Im}\lambda)^{-1}\|, \tag{4.17}
$$

$$
\|G_n(\lambda)^{-1}\| \leq (\|\lambda\| + K)^2\|(\mathrm{Im}\lambda)^{-1}\|, \tag{4.18}
$$

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which hold for arbitrary $\lambda \in M_m(\mathbb{C})$ with $\text{Im}\lambda > 0$ (cf. [HT2, Lemma 3.1], (3.5), the remarks preceeding [HT2, Lemma 5.4], and (2.11)). By Theorem 2.4

$$\left\| \Lambda_n(\lambda) - \lambda + \frac{1}{n} R_n(\lambda) G_n(\lambda)^{-1} \right\| \leq \frac{C_1}{n^2} (\|\lambda\| + K)^2 \|(\text{Im}\lambda)^{-1}\|^5, \qquad (4.19)$$

and according to Remark 4.2 and (4.17) we have

$$\frac{1}{n} \left\| (R_n(\lambda) - R(\lambda)) G(\lambda)^{-1} \right\| \leq \frac{2304 \, \tilde{C}_2 m^2}{n^2} \Big(\sum_{j=1}^{r} \|a_j\|^2 \Big) (\|\lambda\| + K)^4 (\|(\text{Im}\lambda)^{-1}\|^9 + \|(\text{Im}\lambda)^{-1}\|^7). \qquad (4.20)$$

By (4.8)

$$\|R_n(\lambda)\| \leq 2m^2 \Big(\sum_{j=1}^{r} \|a_j\|^2 \Big) \|(\text{Im}\lambda)^{-1}\|^2,$$

and from Theorem 3.2, (4.17) and (4.18) we have

$$\begin{aligned} \|G_n(\lambda)^{-1} - G(\lambda)^{-1}\| &= \|G_n(\lambda)^{-1}(G(\lambda) - G_n(\lambda))G(\lambda)^{-1}\| \\ &\leq \|G_n(\lambda)^{-1}\| \|G_n(\lambda) - G(\lambda)\| \|G(\lambda)^{-1}\| \\ &\leq \frac{C_2}{n} (\|\lambda\| + K)^6 (\|(\text{Im}\lambda)^{-1}\|^9 + \|(\text{Im}\lambda)^{-1}\|^7). \end{aligned}$$

Therefore

$$\frac{1}{n} \left\| R_n(\lambda)(G_n(\lambda)^{-1} - G(\lambda)^{-1}) \right\| \leq \frac{2C_2 m^2}{n^2} \Big(\sum_{j=1}^{r} \|a_j\|^2 \Big) (\|\lambda\| + K)^8 (\|(\text{Im}\lambda)^{-1}\|^{11} + \|(\text{Im}\lambda)^{-1}\|^9). \qquad (4.21)$$

From (4.19), (4.20) and (4.21) it follows that there is a polynomial P_1 of degree 11 with non-negative coefficients depending only on m, $\|a_0\|, \ldots, \|a_{r-1}\|$ and $\|a_r\|$, such that for all $\lambda \in \mathcal{O}'_n$

$$\left\| \Lambda_n(\lambda) - \lambda + \frac{1}{n} R(\lambda) G(\lambda)^{-1} \right\| \leq \frac{1}{n^2} (\|\lambda\| + K + 1)^8 \cdot P_1(\|(\text{Im}\lambda)^{-1}\|). \qquad (4.22)$$

We proceed as follows: By (4.13)

$$L(\lambda) = (\text{id}_m \otimes \tau)[(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}(R(\lambda)G(\lambda)^{-1} \otimes \mathbf{1}_{\mathcal{A}})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}].$$

Moreover,

$$(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} - (\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} = (\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}[(\Lambda_n(\lambda) - \lambda) \otimes \mathbf{1}_{\mathcal{A}}](\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}. \qquad (4.23)$$

Hence, by Proposition 3.1,

$$\left\|G(\lambda) - G_n(\lambda) + \frac{1}{n}L(\lambda)\right\|$$

$$= \left\|G(\lambda) - G(\Lambda_n(\lambda)) + \frac{1}{n}L(\lambda)\right\|$$

$$\leq \left\|(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} - (\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} + \frac{1}{n}(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}(R(\lambda)G(\lambda)^{-1} \otimes \mathbf{1}_{\mathcal{A}})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\right\|$$

$$\leq \left\|(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}[(\Lambda_n(\lambda) - \lambda + \frac{1}{n}R(\lambda)G(\lambda)^{-1}) \otimes \mathbf{1}_{\mathcal{A}}](\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\right\|$$

$$+ \left\|(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}(\frac{1}{n}R(\lambda)G(\lambda)^{-1} \otimes \mathbf{1}_{\mathcal{A}})((\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} - (\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1})\right\|$$

$$\leq \|(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|\|(\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|\left\|\Lambda_n(\lambda) - \lambda + \frac{1}{n}R(\lambda)G(\lambda)^{-1}\right\|$$

$$+ \frac{1}{n}\|(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|\|R(\lambda)G(\lambda)^{-1}\|\|(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} - (\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|$$

$$\tag{4.24}$$

From Corollary 2.5 and from (4.23) we have

$$\|(\lambda \otimes \mathbf{1} - s)^{-1} - (\Lambda_n(\lambda) \otimes \mathbf{1} - s)^{-1}\| \leq 2\|(\mathrm{Im}\lambda)^{-1}\|^2\|\lambda - \Lambda_n(\lambda)\|$$

$$\leq \frac{2C_1'}{n}(\|\lambda\| + K)^2(\|(\mathrm{Im}\lambda)^{-1}\|^7 + \|(\mathrm{Im}\lambda)^{-1}\|^5).$$

$$\tag{4.25}$$

By insertion of (4.7), (4.15), (4.17) and (4.25) into (4.24) we find that

$$\left\|G(\lambda) - G_n(\lambda) + \frac{1}{n}L(\lambda)\right\| \leq \frac{2}{n^2}(\|\lambda\| + K + 1)^8 P_1(\|(\mathrm{Im}\lambda)^{-1}\|) \|(\mathrm{Im}\lambda)^{-1}\|^2 +$$

$$\frac{4C_1'm^2}{n^2}\left(\sum_{j=1}^r \|a_j\|^2\right)(\|\lambda\| + K')^2(\|(\mathrm{Im}\lambda)^{-1}\|^7 + \|(\mathrm{Im}\lambda)^{-1}\|^5)\|(\mathrm{Im}\lambda)^{-1}\|^4.$$

Since $K' = K + 1 \geq 1$, (4.14) readily follows from this inequality in the case $\lambda \in \mathcal{O}_n'$.

Next, assume that $\lambda \in \mathcal{O} \setminus \mathcal{O}_n'$, i.e.

$$\frac{1}{2} \leq \frac{C_1'}{n}(\|\lambda\| + K)^2(\|(\mathrm{Im}\lambda)^{-1}\|^6 + \|(\mathrm{Im}\lambda)^{-1}\|^4).$$

Then, clearly

$$\left\|G(\lambda) - G_n(\lambda) + \frac{1}{n}L(\lambda)\right\|$$

$$\leq \|G(\lambda) - G_n(\lambda)\| + \frac{1}{n}\|L(\lambda)\|$$

$$\leq \frac{2C_1'}{n}(\|\lambda\| + K)^2(\|(\mathrm{Im}\lambda)^{-1}\|^6 + \|(\mathrm{Im}\lambda)^{-1}\|^4) \cdot \left(\|G(\lambda) - G_n(\lambda)\| + \frac{1}{n}\|L(\lambda)\|\right),$$

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where
$$\|G(\lambda) - G_n(\lambda)\| \leq \frac{C_2}{n}(\|\lambda\| + K)^2(\|(\mathrm{Im}\lambda)^{-1}\|^7 + \|(\mathrm{Im}\lambda)^{-1}\|^5),$$

and

$$\|L(\lambda)\| \leq \|R(\lambda)\|\|G(\lambda)^{-1}\|\|(\mathrm{Im}\lambda)^{-1}\|^2 \leq 2m^2\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)(\|\lambda\| + K)^2\|(\mathrm{Im}\lambda)^{-1}\|^5 \quad (4.26)$$

Since $K' \geq 1$, it is not hard to see from the estimates above that it is possible to choose a polynomial P having the properties stated in the theorem, such that (4.14) holds in both of the cases $\lambda \in \mathcal{O}'_n$ and $\lambda \in \mathcal{O} \setminus \mathcal{O}'_n$. $\blacksquare$

4.5 Theorem. *For $m \in \mathbb{N}$ put*

$$\mathcal{O}^{(-)} = \{\lambda \in M_m(\mathbb{C}) \,|\, \mathrm{Im}\lambda < 0\},$$

and for $\lambda \in \mathcal{O}^{(-)}$ define $G_n(\lambda)$, $G(\lambda)$, $R(\lambda)$ and $L(\lambda)$ by the formulas (4.9), (4.10), (4.12) and (4.13), respectively. Then these four functions taking values in $M_m(\mathbb{C})$ are well-defined and analytic in $\mathcal{O}^{(-)}$. Moreover, (4.14) holds for $\lambda \in \mathcal{O}^{(-)}$ too, with the same polynomial P as in Theorem 4.4.

Proof. For $\mu \in \mathcal{O}$, let $G_n^{(-)}(\mu)$, $G^{(-)}(\mu)$, $R^{(-)}(\mu)$ and $L^{(-)}(\mu)$ be the functions obtained by replacement of S_n and s by $-S_n$ and $-s$ respectively in the formulas (4.9), (4.10), (4.12) and (4.13). Then, for $\lambda \in \mathcal{O}^{(-)}$ one has that $-\lambda \in \mathcal{O}$, and

$$
\begin{align}
G(\lambda) &= -G^{(-)}(-\lambda), & (4.27)\\
G_n(\lambda) &= -G_n^{(-)}(-\lambda), & (4.28)\\
R(\lambda) &= R^{(-)}(-\lambda), & (4.29)\\
L(\lambda) &= -L^{(-)}(-\lambda). & (4.30)
\end{align}
$$

In particular, $G_n(\lambda)$, $G(\lambda)$, $R(\lambda)$ and $L(\lambda)$ are well-defined and analytic in $\mathcal{O}^{(-)}$. Moreover, applying (4.14) with $-S_n$ and $-s$ replacing S_n and s, respectively, we find that for $\mu \in \mathcal{O}$

$$\Big\|G^{(-)}(\mu) - G_n^{(-)}(\mu) + \frac{1}{n}L^{(-)}(\mu)\Big\| \leq \frac{1}{n^2}(\|\mu\| + K')^8 P(\|(\mathrm{Im}\mu)^{-1}\|).$$

For $\lambda \in \mathcal{O}^{(-)}$ we put $\mu = -\lambda \in \mathcal{O}$ and deduce from (4.27), (4.28), (4.29), (4.30) and the estimate above that

$$\Big\|G(\lambda) - G_n(\lambda) + \frac{1}{n}L(\lambda)\Big\| \leq \frac{1}{n^2}(\|\lambda\| + K')^8 P(\|(\mathrm{Im}\lambda)^{-1}\|). \quad \blacksquare$$

5 The spectrum of S_n.

We begin this section with a proof of

5.1 Proposition. *Let $(\mathcal{A}, \tau)$ be a C^*-probability space where τ is a faithful state on $\mathcal{A}$. Let $r \in \mathbb{N}$, and let $\{y_1, \ldots, y_r\}$ be a circular system in $(\mathcal{A}, \tau)$. For $m \in \mathbb{N}$ and $a_0, a_1, \ldots, a_r \in M_m(\mathbb{C})$ with $a_0^* = a_0$ define*

$$
s \;=\; a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=0}^{r} (a_j \otimes y_j + a_j^* \otimes y_j^*),
$$

$$
\overline{s} \;=\; \overline{a_0} \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=0}^{r} (\overline{a_j} \otimes y_j + \overline{a_j}^* \otimes y_j^*),
$$

where $\overline{a_j} = (a_j^*)^t$. Then

$$
\sigma(s) = \sigma(\overline{s}).
$$

5.2 Lemma. *(i) Let $\{x_1, \ldots, x_s\}$ be a semicircular system w.r.t. a faithful state τ. Then there exists a conjugate linear $*$-isomorphism π of $C^*(x_1, \ldots, x_s, \mathbf{1})$ onto itself, such that*

$$
\pi(x_j) = x_j, \qquad (1 \le j \le s),
$$

and $\tau \circ \pi = \overline{\tau}$.

(ii) Let $\{y_1, \ldots, y_r\}$ be a circular system w.r.t. a faithful state τ. Then there exists a conjugate linear $*$-isomorphism ρ of $C^*(y_1, \ldots, y_r, \mathbf{1})$ onto itself, such that

$$
\rho(y_j) = y_j, \qquad (1 \le j \le r),
$$

and $\tau \circ \rho = \overline{\tau}$.

Proof. (i) With $\mathcal{A} = C^*(x_1, \ldots, x_s, \mathbf{1})$, let $\mathcal{A}^c$ be the C^*-algebra obtained from $\mathcal{A}$ in the following way: As a Banach $*$-algebra over the reals, $\mathcal{A}^c$ is identical to $\mathcal{A}$, but multiplication by complex scalars is changed into multiplication by the complex conjugate scalars.

For $a \in \mathcal{A}$ we let a^c denote the corresponding element in $\mathcal{A}^c$, and we define a faithful state τ^c on $\mathcal{A}^c$ by:

$$
\tau^c(a^c) = \overline{\tau(a)}, \qquad (a \in \mathcal{A}).
$$

As all of the mixed moments of $x_1, \ldots, x_s$ are real numbers, the $*$-distribution of $\{x_1^c, \ldots, x_s^c\}$ w.r.t. τ^c is the same as the $*$-distribution of $\{x_1, \ldots, x_s\}$ w.r.t. τ. Furthermore, it is clear that $\mathbf{1}, x_1^c, \ldots, x_s^c$ generate $\mathcal{A}^c$. Hence, by [V1, Remark 1.8] there is a unique $*$-isomorphism ϕ of $\mathcal{A}$ onto $\mathcal{A}^c$ such that

$$
\phi(x_j) = x_j^c, \qquad (1 \le j \le s),
$$

and $\tau = \tau^c \circ \phi$.

Define $\chi : \mathcal{A}^c \to \mathcal{A}$ by

$$
\chi(a^c) = a, \qquad (a \in \mathcal{A}),
$$

and put $\pi = \chi \circ \phi : \mathcal{A} \to \mathcal{A}$. Then π has the properties stated in (i).

(ii) By definition there is a semicircular system $\{x_1, \ldots, x_{2r}\}$ such that

$$y_j = \frac{x_j + \mathrm{i}x_{j+r}}{\sqrt{2}}, \qquad (1 \le j \le r).$$

It is clear that $\mathbf{1}, x_1, \ldots, x_{2r}$ generate the same C^*-algebra $\mathcal{A}$ as $\mathbf{1}, y_1, \ldots, y_r$ do. By (i) there is a conjugate linear $*$-isomorphism π of $\mathcal{A}$ onto $\mathcal{A}$ such that

$$\pi(x_j) = x_j, \qquad (1 \le j \le 2r),$$

and $\tau \circ \pi = \overline{\tau}$.

Now, $\{x_1, \ldots, x_r, -x_{r+1}, \ldots, -x_{2r}\}$ is also a semicircular system which, together with $\mathbf{1}$, generates $\mathcal{A}$. And then again, by [V1, Remark 1.8] there is a unique $*$-automorphism ψ of $\mathcal{A}$, such that

$$\begin{aligned} \psi(x_j) &= x_j, & (1 \le j \le r), \\ \psi(x_{r+j}) &= -x_{r+j}, & (1 \le j \le r), \end{aligned}$$

and $\tau \circ \psi = \tau$. Finally, let $\rho = \psi \circ \pi$. Then ρ has the properties stated in (ii). ∎

Proof of Proposition 5.1. Put $\mathcal{A}_0 = C^*(y_1, \ldots, y_r, \mathbf{1}_{\mathcal{A}})$, and let $\rho : \mathcal{A}_0 \to \mathcal{A}_0$ be the conjugate linear $*$-isomorphism provided by Lemma 5.2 (ii). ρ extends to a conjugate linear $*$-isomorphism

$$\rho_m : M_m(\mathbb{C}) \otimes \mathcal{A}_0 \to M_m(\mathbb{C}) \otimes \mathcal{A}_0$$

uniquely determined by

$$\rho_m(x \otimes y) = \overline{x} \otimes \rho(y), \qquad (x \in M_m(\mathbb{C}), \ y \in \mathcal{A}_0).$$

It is standard to check that for any $v \in M_m(\mathbb{C}) \otimes \mathcal{A}_0$, $\sigma(\rho_m(v)) = \overline{\sigma(v)}$. In particular, as s is self-adjoint,

$$\sigma(\overline{s}) = \sigma(\rho_m(s)) = \overline{\sigma(s)} = \sigma(s). \qquad ∎$$

Let S_n, s, and L be as defined in Theorem 4.4, and for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ define

$$\begin{aligned} g_n(\lambda) &= \mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_n - S_n)^{-1}]\}, \\ g(\lambda) &= (\mathrm{tr}_m \otimes \tau)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]\}, \\ l(\lambda) &= \mathrm{tr}_m(L(\lambda \mathbf{1}_m)). \end{aligned}$$

By Theorem 4.4 and Theorem 4.5 these are actually well-defined.

5.3 Lemma. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let $n \in \mathbb{N}$. Then, with P and K' as in Theorem 4.4,*

$$\left| g(\lambda) - g_n(\lambda) + \frac{1}{n} l(\lambda) \right| \le \frac{1}{n^2} (|\lambda| + K')^8 P(|\mathrm{Im}\lambda|^{-1}). \tag{5.1}$$

Proof. This is an easy consequence of Theorem 4.4 and Theorem 4.5, because

$$g_n(\lambda) = \mathrm{tr}_m(G_n(\lambda \mathbf{1}_m)),$$

and

$$g(\lambda) = \mathrm{tr}_m(G(\lambda \mathbf{1}_m)). \qquad \blacksquare$$

Recall from [Ru, Definition 6.7] that a distribution on $\mathbb{R}$ is a linear functional, $\Lambda : C_c^\infty(\mathbb{R}) \to \mathbb{C}$, which is continuous w.r.t. a certain topology on $C_c^\infty(\mathbb{R})$. According to [Ru, Theorem 6.23], for any such Λ there exists a smallest closed set, $F \subseteq \mathbb{R}$, such that for all $\phi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\phi) \cap F = \emptyset$, $\Lambda(\phi) = 0$. F is called the support of Λ and is denoted by $\mathrm{supp}(\Lambda)$. We denote by $\mathcal{D}'_c(\mathbb{R})$ the set of compactly supported distributions on $\mathbb{R}$. By [Ru, Theorem 6.24 (d)], every $\Lambda \in \mathcal{D}'_c(\mathbb{R})$ has a natural extension to a linear funtional on $C^\infty(\mathbb{R})$. We let Λ denote this extension as well.

In the proof of Lemma 5.5 we shall need the following result proved by Tillmann in 1953:

5.4 Theorem. [Ti, Satz 9 & 10]

(i) *Let Λ be a distribution on $\mathbb{R}$ with compact support. Define the Stieltjes transform of Λ, $l : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$, by*

$$l(\lambda) = \Lambda\Big(\frac{1}{\lambda - x}\Big), \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}).$$

Then l is analytic in $\mathbb{C} \setminus \mathbb{R}$ and has an analytic continuation to $\mathbb{C} \setminus \mathrm{supp}(\Lambda)$. Moreover,

(a) *$l(\lambda) \to 0$, as $|\lambda| \to \infty$,*

(b) *there exist a constant $C > 0$, an $n \in \mathbb{N}$ and a compact set $K \subset \mathbb{R}$ containing $\mathrm{supp}(\Lambda)$, such that*

$$|l(\lambda)| \le C \cdot \max\{\mathrm{dist}(\lambda, K)^{-n}, 1\}, \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}),$$

and

(c) *for any $\phi \in C_c^\infty(\mathbb{R})$*

$$\Lambda(\phi) = \lim_{y \to 0^+} \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}} \phi(x)[l(x + \mathrm{i}y) - l(x - \mathrm{i}y)]\mathrm{d}x.$$

(ii) *Conversely, if K is a compact subset of $\mathbb{R}$, and if $l : \mathbb{C} \setminus K \to \mathbb{C}$ is an analytic function satisfying (a) and (b) above, then l is the Stieltjes transform of a compactly supported distribution Λ on $\mathbb{R}$. Moreover, $\mathrm{supp}(\Lambda)$ is exactly the set of singular points of l in K.*

5.5 Lemma. *There is a distribution $\Lambda \in \mathcal{D}'_c(\mathbb{R})$ with $\mathrm{supp}(\Lambda) \subseteq \sigma(s)$, such that for any $\phi \in C_c^\infty(\mathbb{R})$*

$$\Lambda(\phi) = \lim_{y \to 0^+} \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}} \phi(x)[l(x + \mathrm{i}y) - l(x - \mathrm{i}y)]\mathrm{d}x. \qquad (5.2)$$

Proof. We show that l satisfies (a) and (b) of Theorem 5.4 (i). By Proposition 5.1, $\sigma(\overline{s}) = \sigma(s)$. In addition,

$$L(\lambda \mathbf{1}_m) = (\mathrm{id}_m \otimes \tau)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}(R(\lambda \mathbf{1}_m)G(\lambda \mathbf{1}_m)^{-1} \otimes \mathbf{1}_{\mathcal{A}})(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]$$

where

$$
\begin{aligned}
R(\lambda \mathbf{1}_m) \;=\; & \sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j e_{kl}^{(m)}(\mathrm{id}_m \otimes \tau)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - \overline{s})^{-1}(e_{kl}^{(m)} a_j \otimes \mathbf{1}_{\mathcal{A}})(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}] \\
& + \sum_{j=1}^{r}\sum_{k,l=1}^{m} a_j^* e_{kl}^{(m)}(\mathrm{id}_m \otimes \tau)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - \overline{s})^{-1}(e_{kl}^{(m)} a_j^* \otimes \mathbf{1}_{\mathcal{A}})(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]
\end{aligned}
$$

and

$$G(\lambda \mathbf{1}_m)^{-1} = \lambda \mathbf{1}_m - a_0 - \sum_{j=1}^{r}(a_j G(\lambda \mathbf{1}_m)a_j^* + a_j^* G(\lambda \mathbf{1}_m)a_j).$$

It follows that L and l have analytic continuations to $\mathbb{C} \setminus \sigma(s)$.

Moreover,

$$|l(\lambda)| \le \|L(\lambda \mathbf{1}_m)\| \le \|R(\lambda \mathbf{1}_m)\|\|G(\lambda \mathbf{1}_m)^{-1}\|\|(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|^2,$$

where, for $|\lambda| > \|s\|$,

$$
\begin{aligned}
\|R(\lambda \mathbf{1}_m)\| \;\le\; & 2m^2\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)\|(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|^2 \\
\;\le\; & 2m^2\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)\Big(\frac{1}{|\lambda| - \|s\|}\Big)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\|G(\lambda \mathbf{1}_m)^{-1}\| \;=\; & \Big\|\lambda \mathbf{1}_m - a_0 - \sum_{j=1}^{r}(a_j G(\lambda \mathbf{1}_m)a_j^* + a_j^* G(\lambda \mathbf{1}_m)a_j)\Big\| \\
\;\le\; & |\lambda| + \|a_0\| + 2\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)\frac{1}{|\lambda| - \|s\|}.
\end{aligned}
$$

It follows that

$$|l(\lambda)| \le O\Big(\frac{1}{|\lambda|^3}\Big), \quad \text{as } |\lambda| \to \infty. \tag{5.3}$$

Hence, (a) is satisfied.

By (4.26), for $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$|l(\lambda)| \le \|L(\lambda \mathbf{1}_m)\| \le 2m^2\Big(\sum_{j=1}^{r}\|a_j\|^2\Big)(|\lambda| + K')^2|(\mathrm{Im}\lambda)^{-1}|^5. \tag{5.4}$$

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Now, choose $a, b \in \mathbb{R}$, $a < b$, such that $\sigma(s) \subseteq [a, b]$, put $K = [a - 1, b + 1]$, and put

$$D = \{\lambda \in \mathbb{C} \mid 0 < \text{dist}(\lambda, K) \leq 1\}.$$

Note that for $\lambda \in D$, either $\text{dist}(\lambda, K) = |\text{Im}\lambda|$ or $\text{dist}(\lambda, \sigma(s)) \geq 1$. Hence, by (5.4) and the fact that l is bounded on compact subsets of $\mathbb{C} \setminus \sigma(s)$, we find that for some constant $C < \infty$:

$$\forall \lambda \in D : \quad |l(\lambda)| \leq C \cdot \max\{\text{dist}(\lambda, K)^{-5}, 1\} = C \cdot \text{dist}(\lambda, K)^{-5}.$$

By (5.3) l is bounded on $\mathbb{C} \setminus D$. Hence, if C is chosen sufficiently large, then (b) holds with $K = [a - 1, b + 1]$ and $n = 5$, and the lemma follows from Theorem 5.4. $\blacksquare$

5.6 Theorem. *Let S_n, s, P and K' be as in Theorem 4.4, and let $\Lambda \in \mathcal{D}'_c(\mathbb{R})$ be as in Lemma 5.5. Then for any $\phi \in C_c^\infty(\mathbb{R})$:*

$$\left| \mathbb{E}\{(\text{tr}_m \otimes \text{tr}_n)\phi(S_n)\} - (\text{tr}_m \otimes \tau)\phi(s) - \frac{1}{n}\Lambda(\phi) \right|$$

$$\leq \frac{1}{n^2}\left(\frac{2^7}{13!\pi} \int_{\mathbb{R} \times \mathbb{R}} |((1 + D)^{14}\phi)(x)|(|x| + \sqrt{2}t + K')^8 Q(t)\mathrm{e}^{-t}\mathrm{d}m(x) \otimes \mathrm{d}m(t) \right),$$
(5.5)

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$, and Q is the polynomial of degree less than 13 defined by $Q(t) = P(t^{-1})t^{13}$. In particular,

$$\mathbb{E}\{(\text{tr}_m \otimes \text{tr}_n)\phi(S_n)\} = (\text{tr}_m \otimes \tau)\phi(s) + \frac{1}{n}\Lambda(\phi) + O\left(\frac{1}{n^2}\right).$$
(5.6)

Proof. By the Riesz representation theorem there are unique probability measures μ_n and μ on $(\mathbb{R}, \mathbb{B})$ such that for any $\psi \in C_0(\mathbb{R})$

$$\int_{\mathbb{R}} \psi \, \mathrm{d}\mu_n = \mathbb{E}\{(\text{tr}_m \otimes \text{tr}_n)\psi(S_n)\},$$

and

$$\int_{\mathbb{R}} \psi \, \mathrm{d}\mu = (\text{tr}_m \otimes \tau)\psi(s).$$

In particular, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we have

$$g_n(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x}\mathrm{d}\mu_n(x),$$

and

$$g(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x}\mathrm{d}\mu(x).$$

By the inverse Stieltjes transform

$$\mathrm{d}\mu_n(x) = \lim_{y \to 0^+} \left(-\frac{1}{\pi}\text{Im}(g_n(x + \mathrm{i}y))\mathrm{d}x \right) = \lim_{y \to 0^+} \left(\frac{\mathrm{i}}{2\pi}(g_n(x + \mathrm{i}y) - g_n(x - \mathrm{i}y))\mathrm{d}x \right),$$

and

$$\mathrm{d}\mu(x) = \lim_{y\to 0^+}\left(-\frac{1}{\pi}\mathrm{Im}(g(x+\mathrm{i}y))\mathrm{d}x\right) = \lim_{y\to 0^+}\left(\frac{\mathrm{i}}{2\pi}(g(x+\mathrm{i}y) - g(x-\mathrm{i}y))\mathrm{d}x\right),$$

in the sense of weak convergence of probability measures on $(\mathbb{R}, \mathbb{B})$. Combining these observations with Lemma 5.5 we find that for any $\phi \in C_c^\infty(\mathbb{R})$

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(S_n)\} - (\mathrm{tr}_m \otimes \tau)\phi(s) - \frac{1}{n}\Lambda(\phi) =$$

$$\lim_{y\to 0^+}\frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}}\phi(x)[g_n(x+\mathrm{i}y) - g(x+\mathrm{i}y) - \frac{1}{n}l(x+\mathrm{i}y)-$$

$$(g_n(x-\mathrm{i}y) - g(x-\mathrm{i}y) - \frac{1}{n}l(x-\mathrm{i}y))]\mathrm{d}x.$$

Thus, with

$$r_n(\lambda) = g_n(\lambda) - g(\lambda) - \frac{1}{n}l(\lambda), \qquad (\lambda \in \mathbb{C}\setminus\mathbb{R}),$$

$$\left|\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(S_n)\} - (\mathrm{tr}_m \otimes \tau)\phi(s) - \frac{1}{n}\Lambda(\phi)\right|$$

$$\leq \frac{1}{2\pi}\limsup_{y\to 0^+}\left(\left|\int_{\mathbb{R}}\phi(x)r_n(x+\mathrm{i}y)\mathrm{d}x\right| + \left|\int_{\mathbb{R}}\phi(x)r_n(x-\mathrm{i}y)\mathrm{d}x\right|\right). \tag{5.7}$$

The rest of the proof follows the lines of [HT2, Proof of Theorem 6.2]. For $\mathrm{Im}\lambda \neq 0$ and $p \in \mathbb{N}$ define

$$I_p(\lambda) = \frac{1}{(p-1)!}\int_{\mathbb{R}}r_n(\lambda + t)t^{p-1}\mathrm{e}^{-t}\mathrm{d}t.$$

By (5.1), if $\varepsilon > 0$ and $|\mathrm{Im}\lambda| > \varepsilon$, then for every $t > 0$

$$|r_n(\lambda + t)| \leq \frac{1}{n^2}(|\lambda + t| + K')^6 P(|\mathrm{Im}(\lambda + t)|^{-1}) \leq \frac{1}{n^2}(|\lambda| + t + K')^8 P(\varepsilon^{-1}).$$

Therefore $I_p(\lambda)$ is well-defined, and it is standard to check that I_p is analytic.

Now, consider a fixed $\phi \in C_c^\infty(\mathbb{R})$, and let $y > 0$. After p steps of integration by parts we obtain:

$$\int_{\mathbb{R}}\phi(x)r_n(x\pm\mathrm{i}y)\mathrm{d}x = \int_{\mathbb{R}}((1+D)^p\phi)(x)I_p(x\pm\mathrm{i}y)\mathrm{d}x,$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$.

We are going to estimate $|I_{14}(\lambda)|$. Let's start with the case $\mathrm{Im}\lambda > 0$ and define

$$F(z) = \frac{1}{13!}r_n(\lambda + z)z^{13}\mathrm{e}^{-z}, \qquad (\mathrm{Im}z > -\mathrm{Im}\lambda).$$

F is analytic, so for any $r > 0$

$$\int_{[0,r]}F(z)\mathrm{d}z + \int_{[r,r+\mathrm{i}r]}F(z)\mathrm{d}z + \int_{[r+\mathrm{i}r,0]}F(z)\mathrm{d}z = 0.$$

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The second term in the expression above tends to zero, as r goes to infinity. Indeed,

$$\left| \int_{[r,r+\mathrm{i}r]} F(z)\mathrm{d}z \right| = \left| \int_0^r F(r+\mathrm{i}t)\mathrm{d}t \right|$$

$$\leq \frac{2^{13}}{13!} \int_0^r |r_n(\lambda + r + \mathrm{i}t)| r^{13}\mathrm{e}^{-r}\mathrm{d}t$$

$$\leq \frac{2^{13}}{13!} \int_0^r \left(|g_n(\lambda + r + \mathrm{i}t)| + |g(\lambda + r + \mathrm{i}t)| + \frac{1}{n}|l(\lambda + r + \mathrm{i}t)| \right) r^{13}\mathrm{e}^{-r}\mathrm{d}t$$

$$\leq \frac{2^{13}}{13!} \int_0^r \left(\frac{2}{|\mathrm{Im}\lambda + t|} + \frac{1}{n}2m^2(\sum_{j=1}^r \|a_j\|^2)(|\lambda + r + \mathrm{i}t| + K)^2 \frac{1}{|\mathrm{Im}\lambda + t|^5} \right) r^{13}\mathrm{e}^{-r}\mathrm{d}t$$

$$\leq \frac{2^{13}}{13!} \left(\frac{2}{|\mathrm{Im}\lambda|} + \frac{1}{n}2m^2(\sum_{j=1}^r \|a_j\|^2)(|\lambda| + 2r + K)^2 \frac{1}{|\mathrm{Im}\lambda|^5} \right) r^{14}\mathrm{e}^{-r}$$

$$\to 0, \quad \text{as } r \to \infty.$$

Consequently,

$$
\begin{aligned}
I_{14}(\lambda) &= \lim_{r\to\infty} \int_{[0,r+\mathrm{i}r]} F(z)\mathrm{d}z \\
&= \int_0^\infty F((1+\mathrm{i})t)(1+\mathrm{i})\mathrm{d}t \\
&= \frac{1}{13!} \int_0^\infty r_n(\lambda + (1+\mathrm{i})t)(1+\mathrm{i})^{14} t^{13} \mathrm{e}^{-(1+\mathrm{i})t}\mathrm{d}t,
\end{aligned}
$$

and by (5.1)

$$
\begin{aligned}
|I_{14}(\lambda)| &\leq \frac{2^7}{13!n^2} \int_0^\infty (|\lambda + (1+\mathrm{i})t| + K')^8 P(|\mathrm{Im}\lambda + t|^{-1}) t^{13}\mathrm{e}^{-t}\mathrm{d}t \\
&\leq \frac{2^7}{13!n^2} \int_0^\infty (|\lambda| + \sqrt{2}t + K')^8 P(t^{-1}) t^{13}\mathrm{e}^{-t}\mathrm{d}t \\
&= \frac{2^7}{13!n^2} \int_0^\infty (|\lambda| + \sqrt{2}t + K')^8 Q(t)\mathrm{e}^{-t}\mathrm{d}t
\end{aligned}
$$

with $Q(t) = P(t^{-1})t^{13}$. The case $\mathrm{Im}\lambda < 0$ is treated similarly, and we obtain the same upper bound as the one above. Thus, for every $y > 0$

$$\left| \int_{\mathbb{R}} \phi(x)r_n(x \pm \mathrm{i}y)\mathrm{d}x \right|$$

$$\leq \frac{1}{n^2} \left(\frac{2^7}{13!} \int_{\mathbb{R}\times\mathbb{R}} |((1+D)^{14}\phi)(x)|(|x \pm \mathrm{i}y| + \sqrt{2}t + K')^8 Q(t)\mathrm{e}^{-t}\mathrm{d}m(x) \otimes \mathrm{d}m(t) \right),$$

and by dominated convergence,

$$\limsup_{y\to 0^+} \left| \int_{\mathbb{R}} \phi(x)r_n(x \pm \mathrm{i}y)\mathrm{d}x \right|$$

$$\leq \frac{1}{n^2} \left(\frac{2^7}{13!} \int_{\mathbb{R}\times\mathbb{R}} |((1+D)^{14}\phi)(x)|(|x| + \sqrt{2}t + K')^8 Q(t)\mathrm{e}^{-t}\mathrm{d}m(x) \otimes \mathrm{d}m(t) \right).$$

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By (5.7) this completes the proof. ∎

An application of Remark 2.2 and Lemma 2.3 yields (as in [HT2, Proof of Proposition 4.7]) the following estimate:

5.7 Proposition. *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a C^1-function with compact support. Then*

$$\mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n)\} \leq \frac{4}{n^2}\Big(\sum_{j=1}^{r} \|a_j\|^2\Big) \mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)|\psi'|^2(S_n)\}.$$

Finally, as in [HT2, Proof of Lemma 6.3] and [HT2, Proof of Theorem 6.4], we may combine Theorem 5.6 and Proposition 5.7 to get:

5.8 Theorem. *Let $m \in \mathbb{N}$, let $a_0, \ldots, a_r \in M_m(\mathbb{C})$ with $a_0^* = a_0$, and let S_n and s be as defined in Theorem 4.4. Then for any $\varepsilon > 0$ and for almost all $\omega \in \Omega$,*

$$\sigma(S_n(\omega)) \subseteq \sigma(s) + (-\varepsilon, \varepsilon),$$

eventually as $n \to \infty$.

5.9 Remark. Let r and s be non-negative integers with $r + s \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ be stochastically independent random matrices such that $X_1^{(n)}, \ldots, X_r^{(n)} \in$ GOE$(n, \frac{1}{n})$ and $X_{r+1}^{(n)}, \ldots, X_{r+s}^{(n)} \in$ GOE$^*(n, \frac{1}{n})$. Furthermore, let $a_0, b_1, \ldots, b_{r+s} \in M_m(\mathbb{C})_{sa}$, and put

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{j=1}^{r+s} b_j \otimes X_j^{(n)}.$$

For $\{x_1, \ldots, x_{r+s}\}$ a semicircular system in $(\mathcal{A}, \tau)$ put

$$s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r+s} b_j \otimes x_j.$$

As mentioned in the introduction, the $X_j^{(n)}$'s may be expressed in terms of $r + s$ stochastically independent random matrices $Y_1^{(n)}, \ldots, Y_{r+s}^{(n)} \in$ GRM$^{\mathbb{R}}(n, \frac{1}{n})$, and for suitable $a_1, \ldots, a_r \in M_m(\mathbb{C})$

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{j=1}^{r} (a_j \otimes Y_j^{(n)} + a_j^* \otimes Y_j^{(n)*}).$$

We may also assume that there is a circular system $\{y_1, \ldots, y_{r+s}\}$ in $(\mathcal{A}, \tau)$ such that

$$x_j = \frac{1}{\sqrt{2}}(y_j + y_j^*), \qquad (1 \leq j \leq r),$$

$$x_j = \frac{1}{i\sqrt{2}}(y_j - y_j^*), \qquad (r+1 \leq j \leq r+s).$$

Then, with $a_1, \ldots, a_r$ as above,

$$s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r+s} (a_j \otimes y_j + a_j^* \otimes y_j^*),$$

and it follows from Theorem 5.8 that for any $\varepsilon > 0$

$$\sigma(S_n(\omega)) \subseteq \sigma(s) + (-\varepsilon, \varepsilon),$$

eventually as $n \to \infty$ for almost all $\omega \in \Omega$. Thus, by the proof of [HT2, Proposition 7.3] we have:

5.10 Proposition. *Let r and s be non-negative integers with $r + s \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ be stochastically independent random matrices such that $X_1^{(n)}, \ldots, X_r^{(n)} \in \mathrm{GOE}(n, \frac{1}{n})$ and $X_{r+1}^{(n)}, \ldots, X_{r+s}^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{n})$. Furthermore, let $\{x_1, \ldots, x_{r+s}\}$ be a semicircular system in a C^*-probability space $(\mathcal{A}, \tau)$ with τ a faithful state on $\mathcal{A}$. Then there is a P-null set $N' \subseteq \Omega$ such that for any non-commutative polynomial p in $r + s$ variables and for every $\omega \in \Omega \setminus N'$:*

$$\limsup_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))\| \leq \|p(x_1, \ldots, x_{r+s})\|. \tag{5.8}$$

6 Almost sure convergence of mixed moments.

Throughout this section let r and s be fixed numbers in $\mathbb{N}_0$ with $r + s \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ be stochastically independent random matrices such that $X_1^{(n)}, \ldots, X_r^{(n)} \in \mathrm{GOE}(n, \frac{1}{n})$ and $X_{r+1}^{(n)}, \ldots, X_{r+s}^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{n})$.

Also, let $(\mathcal{A}, \tau)$ denote a C^*-probability space, and let $x_1, \ldots, x_{r+s}$ be a semicircular system in $(\mathcal{A}, \tau)$.

The aim of this section is to prove:

6.1 Proposition. *There is a P-null set $N'' \subseteq \Omega$ such that for any non-commutative polynomial p in $r + s$ variables and for every $\omega \in \Omega \setminus N''$:*

$$\liminf_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))\| \geq \|p(x_1, \ldots, x_{r+s})\|. \tag{6.1}$$

From [HT2, Proof of Lemma 7.2] it is clear that this proposition follows from the following theorem:

6.2 Theorem. *For every $p \in \mathbb{C}\langle X_1, \ldots, X_{r+s}\rangle$*

$$\lim_{n \to \infty} \mathrm{tr}_n(p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))) = \tau(p(x_1, \ldots, x_{r+s})) \tag{6.2}$$

holds for almost all $\omega \in \Omega$.

Note that Theorem 6.2 is the strong version of the following result due to Voiculescu:

6.3 Theorem. [V2, Theorem 2.3] Let r, s and $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ be as defined above. Then

(a) the sets $\{X_j^{(n)}\}_{1 \leq j \leq r+s}$ are asymptotically free as $n \to \infty$, and

(b) for all $j \in \{1, \ldots, r+s\}$ and for all $k \in \mathbb{N}$

$$\lim_{n \to \infty} \mathbb{E}\{\mathrm{tr}_n((X_j^{(n)})^k)\} = \frac{1}{2\pi} \int_{-2}^{2} x^k \sqrt{4 - x^2} dx. \tag{6.3}$$

In particular, for any $p \in \mathbb{C}\langle X_1, \ldots, X_{r+s}\rangle$

$$\lim_{n \to \infty} \mathbb{E}\{\mathrm{tr}_n(p(X_1^{(n)}, \ldots, X_{r+s}^{(n)}))\} = \tau(p(x_1, \ldots, x_{r+s})). \tag{6.4}$$

Actually [V2, Theorem 2.3] is slightly different from Theorem 6.3 above, since in [V2] the diagonal elements of the first r random matrices $X_1^{(n)}, \ldots, X_r^{(n)}$ have variance $\frac{1}{n}$ and not $\frac{2}{n}$ as in the definition of $\mathrm{GOE}(n, \frac{1}{n})$. However, when one works out the details of the proof sketched in [V2], it is easily seen that this change of the diagonal entries does not affect the validity of the result.

In order to prove that Theorem 6.3 implies Theorem 6.2 we shall make use of the following two lemmas:

6.4 Lemma. Let $k \in \mathbb{N}$. Then there is a constant $C_1(k) < \infty$ such that for all $n \in \mathbb{N}$ and for all $X^{(n)} \in SGRM(n, \frac{1}{n})$

$$\mathbb{E}\{\|X^{(n)}\|^k\} \leq C_1(k). \tag{6.5}$$

Proof. According to [HT2, Lemma 5.1] the constant 4 dominates $\mathbb{E}\{\|X^{(n)}\|\}$ for all $n \in \mathbb{N}$. Hence, we shall concentrate on the case $k \geq 2$. Let $n \in \mathbb{N}$, and let $X^{(n)} \in SGRM(n, \frac{1}{n})$. Let $\lambda_{\max}(X^{(n)})$ (respectively $\lambda_{\min}(X^{(n)})$) denote the largest (respectively the smallest) eigenvalue of $X^{(n)}$. Then for all $\varepsilon > 0$ we have (cf. [HT1, Proof of Lemma 3.3])

$$P(\lambda_{\max}(X^{(n)}) > 2 + \varepsilon) \leq n \cdot \exp\left(-\frac{n\varepsilon^2}{2}\right),$$

$$P(\lambda_{\min}(X^{(n)}) < -(2 + \varepsilon)) \leq n \cdot \exp\left(-\frac{n\varepsilon^2}{2}\right),$$

where the last equality follows from the first and the fact that $-X^{(n)} \in SGRM(n, \frac{1}{n})$. Hence,

$$P(\|X^{(n)}\| > 2 + \varepsilon) \leq 2n \cdot \exp\left(-\frac{n\varepsilon^2}{2}\right). \tag{6.6}$$

Put

$$\varepsilon_0 = \sqrt{\frac{2 \log 2n}{n}}, \tag{6.7}$$

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and note that

$$2n \cdot \exp\left(-\frac{n\varepsilon_0^2}{2}\right) = 1. \tag{6.8}$$

Define $F : \mathbb{R} \to [0, 1]$ by

$$F(t) = P(\|X^{(n)}\| \le t), \quad (t \in \mathbb{R}). \tag{6.9}$$

Then from the estimate (6.6) it follows that for all $\varepsilon > 0$

$$F(2 + \varepsilon) \ge 1 - 2n \cdot \exp\left(-\frac{n\varepsilon^2}{2}\right), \tag{6.10}$$

and carrying out integration by parts (cf. [Fe, Lemma V.6.1]) we obtain

$$
\begin{aligned}
\mathbb{E}\{\|X^{(n)}\|^k\} &= \int_0^\infty t^k \mathrm{d}F(t) \\
&= k \int_0^\infty t^{k-1}(1 - F(t))\mathrm{d}t \\
&= k \int_0^{2+\varepsilon_0} t^{k-1}(1 - F(t))\mathrm{d}t + k \int_{2+\varepsilon_0}^\infty t^{k-1}(1 - F(t))\mathrm{d}t \\
&\le (2 + \varepsilon_0)^k + k \int_{\varepsilon_0}^\infty (2 + t)^{k-1} 2n \exp\left(-\frac{nt^2}{2}\right)\mathrm{d}t.
\end{aligned}
$$

Now, using (6.8) we get

$$
\begin{aligned}
k \int_{\varepsilon_0}^\infty (2 + t)^{k-1} 2n & \exp\left(-\frac{nt^2}{2}\right)\mathrm{d}t \\
&= k \int_{\varepsilon_0}^\infty (2 + t)^{k-1} 2n \exp\left(-\frac{n(t - \varepsilon_0)^2}{2} - \frac{n\varepsilon_0^2}{2} - n\varepsilon_0(t - \varepsilon_0)\right)\mathrm{d}t \\
&\le k\, 2n \exp\left(-\frac{n\varepsilon_0^2}{2}\right) \int_{\varepsilon_0}^\infty (2 + t)^{k-1} \exp\left(-\frac{n(t - \varepsilon_0)^2}{2}\right)\mathrm{d}t \\
&= k \int_0^\infty (2 + t + \varepsilon_0)^{k-1} \exp\left(-\frac{nt^2}{2}\right)\mathrm{d}t \\
&\le k \int_0^\infty (2 + t + \varepsilon_0)^{k-1} \exp\left(-\frac{t^2}{2}\right)\mathrm{d}t.
\end{aligned}
$$

It is easily shown that the function $g : [\frac{1}{2}, \infty[\to \mathbb{R}$ defined by

$$g(x) = \sqrt{\frac{2 \log 2x}{x}}, \quad (x \ge \frac{1}{2}) \tag{6.11}$$

attains its maximum at $x = \mathrm{e}$, and this maximum is $2\mathrm{e}^{-\frac{1}{2}}$ which is less than 2. Hence,

$$\mathbb{E}\{\|X^{(n)}\|^k\} \le 4^k + k \int_0^\infty (4 + t)^{k-1} \exp\left(-\frac{t^2}{2}\right)\mathrm{d}t,$$

and the lemma follows. ∎

6.5 Lemma. *Let $d \in \mathbb{N}$, let $i_1, \ldots, i_d \in \{1, \ldots, r+s\}$, and let $n \in \mathbb{N}$. Define $f : M_n(\mathbb{C})^{r+s} \to \mathbb{C}$ by*

$$f(v_1, \ldots, v_{r+s}) = \mathrm{tr}_n(v_{i_1} \cdot \cdots \cdot v_{i_d}), \quad (v_1, \ldots, v_{r+s} \in M_n(\mathbb{C})). \tag{6.12}$$

Then there is a constant $C_2(d) > 0$ (independent of n) such that

$$\mathbb{V}\{f(X_1^{(n)}, \ldots, X_{r+s}^{(n)})\} \leq \frac{C_2(d)}{n^2}. \tag{6.13}$$

Proof. From arguments similar to those presented in Remark 2.2 it follows that

$$\mathbb{V}\{f(X_1^{(n)}, \ldots, X_{r+s}^{(n)})\} \leq \frac{1}{2n}\mathbb{E}\{\|(\mathrm{grad}f)(X_1^{(n)}, \ldots, X_{r+s}^{(n)})\|_e^2\}. \tag{6.14}$$

Now, let $v = (v_1, \ldots, v_{r+s}) \in M_n(\mathbb{C})^{r+s}$, and let $w = (w_1, \ldots, w_{r+s}) \in M_n(\mathbb{C})^{r+s}$ with $\|w\|_e = 1$. By the Cauchy-Schwartz inequality:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw)\right| = \frac{1}{n}\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Tr}_n((v_{i_1}+tw_{i_1}) \cdot \cdots \cdot (v_{i_d}+tw_{i_d}))\right|$$

$$\leq \frac{1}{n}\|\mathbf{1}_n\|_{2,\mathrm{Tr}_n}(\|w_{i_1}v_{i_2}\cdots v_{i_d}\|_{2,\mathrm{Tr}_n} + \|v_{i_1}w_{i_2}\cdots v_{i_d}\|_{2,\mathrm{Tr}_n} + \cdots + \|v_{i_1}v_{i_2}\cdots w_{i_d}\|_{2,\mathrm{Tr}_n})$$

$$\leq \frac{1}{\sqrt{n}}(\|w_{i_1}\|_{2,\mathrm{Tr}_n}\|v_{i_2}\cdots v_{i_d}\| + \|v_{i_1}\|\|w_{i_2}\|_{2,\mathrm{Tr}_n}\|v_{i_3}\cdots v_{i_d}\| + \cdots + \|v_{i_1}\cdots v_{i_{d-1}}\|\|w_{i_d}\|_{2,\mathrm{Tr}_n})$$

Hence, with $M = \max_{1 \leq j \leq r+s} \|v_j\|$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw)\right| \leq \frac{1}{\sqrt{n}}M^{d-1}\sum_{j=1}^{d}\|w_{i_j}\|_{2,\mathrm{Tr}_n}$$

$$\leq \frac{1}{\sqrt{n}}M^{d-1}d,$$

and since w was arbitrary,

$$\|\mathrm{grad}f\|_e^2 \leq \frac{1}{n}M^{2(d-1)}d^2. \tag{6.15}$$

It follows from (6.14) that

$$\mathbb{V}\{f(X_1^{(n)}, \ldots, X_{r+s}^{(n)})\} \leq \frac{d^2}{2n^2}\mathbb{E}\{(\max_{1 \leq j \leq r+s}\|X_j^{(n)}\|)^{2(d-1)}\}$$

$$\leq \frac{d^2}{2n^2}\sum_{j=1}^{r+s}\mathbb{E}\{\|X_j^{(n)}\|^{2(d-1)}\}.$$

For any $X \in \mathrm{GOE}(n, \frac{1}{n})$ we may choose $Y \in \mathrm{SGRM}(n, \frac{1}{n})$ such that

$$X = \frac{1}{\sqrt{2}}(Y + \overline{Y}), \tag{6.16}$$

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Similarly, when $X \in \text{GOE}^*(n, \frac{1}{n})$ we may choose $Y \in \text{SGRM}(n, \frac{1}{n})$ such that

$$X = \frac{1}{\sqrt{2}}(Y - \overline{Y}). \tag{6.17}$$

Then in both cases we have

$$\mathbb{E}\{\|X\|^{2(d-1)}\} \le 2^{d-1}\mathbb{E}\{\|Y\|^{2(d-1)}\}, \tag{6.18}$$

and applying Lemma 6.4 we obtain the desired estimate with

$$C_2(d) = d^2(r+s)2^{d-2} \cdot C_1(2d-2). \qquad \blacksquare$$

Proof of Theorem 6.2. Let $d \in \mathbb{N}$, and let $i_1, \ldots, i_d \in \{1, \ldots, r+s\}$. For each $n \in \mathbb{N}$ define a complex random variable Z_n by

$$Z_n = \text{tr}_n(X_{i_1}^{(n)} \cdot \cdots \cdot X_{i_d}^{(n)}) - \mathbb{E}\{\text{tr}_n(X_{i_1}^{(n)} \cdot \cdots \cdot X_{i_d}^{(n)})\}. \tag{6.19}$$

By the Borel-Cantelli Lemma, if

$$\sum_{n=1}^{\infty} P(|Z_n| > n^{-1/3}) < \infty, \tag{6.20}$$

then

$$P(|Z_n| \le n^{-1/3}, \text{eventually as } n \to \infty) = 1. \tag{6.21}$$

In particular, the sequence $(Z_n)_{n=1}^{\infty}$ tends to zero almost surely, and it follows from Theorem 6.3 that

$$\lim_{n \to \infty} \text{tr}_n(X_{i_1}^{(n)} \cdots X_{i_d}^{(n)}) = \tau(x_{i_1} \cdots x_{i_d}), \tag{6.22}$$

almost surely. Consequently, Theorem 6.2 holds. To prove (6.20), apply Chebychev's inequality and Lemma 6.5 as follows:

$$
\begin{aligned}
P(|Z_n| > n^{-1/3}) &\le n^{2/3}\mathbb{E}\{|Z_n|^2\} \\
&\le n^{2/3}\frac{C_2(d)}{n^2} \\
&= n^{-4/3}C_2(d). \qquad \blacksquare
\end{aligned}
$$

6.6 Remark. The above proof of Theorem 6.2 follows the main lines of the proof of the corresponding result for the $\text{SGRM}(n, \frac{1}{n})$-case sketched in [Pi, Proof of Theorem 9.9.3].

7   The symplectic case.

We shall use the results of the previous sections to prove:

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7.1 Theorem. *Let r, $s \in \mathbb{N}_0$ with $r + s \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ be stochastically independent random matrices defined on $(\Omega, \mathcal{F}, P)$ such that $X_1^{(n)}, \ldots, X_r^{(n)} \in$ $\mathrm{GSE}(n, \frac{1}{n})$ and $X_{r+1}^{(n)}, \ldots, X_{r+s}^{(n)} \in \mathrm{GSE}^*(n, \frac{1}{n})$. Furthermore, let $(\mathcal{A}, \tau)$ be a C^*-probability space with τ a faithful state on $\mathcal{A}$, and let $\{x_1, \ldots, x_{r+s}\}$ be a semicircular system in $(\mathcal{A}, \tau)$. Then there is a P-null set $N \subseteq \Omega$ such that for any $\omega \in \Omega \setminus N$ and for any polynomial p in $r + s$ non-commuting variables:*

$$\lim_{n \to \infty} \| p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega)) \| = \| p(x_1, \ldots, x_{r+s}) \|.$$

Definitions of the random matrix ensembles $\mathrm{GSE}(n, \frac{1}{n})$ and $\mathrm{GSE}^*(n, \frac{1}{n})$ were given in the Introduction.

7.2 Remark. With $(\mathcal{A}, \tau)$ and $\{x_1, \ldots, x_{r+s}\}$ as in Theorem 7.1, let $(\mathcal{C}, \phi)$ be another C^*-probability space with ϕ a faithful state on $\mathcal{C}$, and let $\{z_1, \ldots, z_{4(r+s)}\}$ be a semicircular system in $(\mathcal{C}, \phi)$. Then, by [VDN, Proposition 5.1.3],

$$\bigcup_{j=1}^{r+s} \left\{ \frac{z_{4j-3} + z_{4j-2}}{\sqrt{2}} \ , \ \frac{z_{4j-3} - z_{4j-2}}{\sqrt{2}} \ , \ z_{4j-1}, \ z_{4j} \right\}$$

is also a semicircular system in $(\mathcal{C}, \phi)$. Hence,

$$\mathcal{M}_1 = \bigcup_{j=1}^{r+s} \left\{ \frac{z_{4j-3} + z_{4j-2}}{\sqrt{2}} \ , \ \frac{z_{4j-3} - z_{4j-2}}{\sqrt{2}} \right\}$$

is a semicircular system,

$$\mathcal{M}_2 = \bigcup_{j=1}^{r+s} \left\{ \frac{z_{4j} - i z_{4j-1}}{\sqrt{2}} \right\}$$

is a circular system, and the sets $\mathcal{M}_1$ and $\mathcal{M}_2$ are $*$-free.

For $1 \leq j \leq r + s$ define $Z_j \in M_2(\mathcal{C})$ by

$$Z_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}(z_{4j-3} - z_{4j-2}) & \frac{1}{\sqrt{2}}(z_{4j} + i z_{4j-1}) \\ \frac{1}{\sqrt{2}}(z_{4j} - i z_{4j-1}) & \frac{1}{\sqrt{2}}(z_{4j-3} + z_{4j-2}) \end{pmatrix}.$$

According to [VDN, Proposition 5.1.6], $\{Z_1, \ldots, Z_{r+s}\}$ is then a semicircular system in $(M_2(\mathcal{C}), \mathrm{tr}_2 \otimes \phi)$. Furthermore, faithfulness of $(\mathrm{tr}_2 \otimes \phi)$ on $M_2(\mathcal{C})$ implies that there is a state-preserving unital $*$-isomorphism $\Phi : C^*(\mathbf{1}_{\mathcal{A}}, x_1, \ldots, x_{r+s}) \to C^*(\mathbf{1}_2 \otimes \mathbf{1}_{\mathcal{A}}, Z_1, \ldots, Z_{r+s})$ such that $\Phi(x_j) = Z_j$, $1 \leq j \leq r + s$ (cf. [V1, Remark 1.8]). Hence, Theorem 7.1 is proved, as soon as we have shown that there is a P-null set $N \subset \Omega$ such that for any $p \in \mathbb{C}\langle X_1, \ldots, X_{r+s} \rangle$ and for every $\omega \in \Omega \setminus N$:

$$\lim_{n \to \infty} \| p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega)) \| = \| p(Z_1, \ldots, Z_{r+s}) \|.$$

7.3 Proposition. *Let t, $u \in \mathbb{N}_0$ with $t + u \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_{t+u}^{(n)}$ be stochastically independent random matrices with $X_1^{(n)}, \ldots, X_t^{(n)} \in \mathrm{GOE}(n, \frac{1}{n})$ and $X_{t+1}^{(n)}, \ldots, X_{t+u}^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{n})$. Furthermore, let $\{x_1, \ldots, x_{t+u}\}$ be a semicircular system in a C^*-probability space $(\mathcal{C}, \phi)$ with ϕ a faithful state on $\mathcal{C}$, and let $\mathcal{B}$ be a unital exact C^*-algebra. Then there is a P-null set $N \subset \Omega$ such that for any polynomial p in $t + u$ non-commuting variables with coefficients in $\mathcal{B}$, and for every $\omega \in \Omega \setminus N$:*

$$\lim_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{t+u}^{(n)}(\omega))\|_{M_n(\mathcal{B})} = \|p(x_1, \ldots, x_{t+u})\|_{\mathcal{B} \otimes_{\min} C^*(\mathcal{C}_0)},$$

where $\mathcal{C}_0 = C^*(\mathbf{1}_{\mathcal{C}}, x_1, \ldots, x_{t+u})$.

Proof. This follows from Theorem A in the same way as [HT2, Theorem 9.1] follows from [HT2, Theorem 7.1]. ∎

Note that Theorem 7.3 applies in the case $\mathcal{B} = M_2(\mathbb{C})$.

Proof of Theorem 7.1 For each $n \in \mathbb{N}$ we may choose stochastically independent random matrices $Z_1^{(n)}, \ldots, Z_{4(r+s)}^{(n)}$ such that for $1 \leq j \leq r$ we have that $Z_{4j-3}^{(n)} \in \mathrm{GOE}(n, \frac{1}{4n})$, $Z_{4j-2}^{(n)}, Z_{4j-1}^{(n)}, Z_{4j}^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{4n})$, and

$$
\begin{aligned}
X_j^{(n)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Z_{4j-3}^{(n)} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes (iZ_{4j-2}^{(n)}) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (iZ_{4j-1}^{(n)}) + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes (iZ_{4j}^{(n)}) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Z_{4j-3}^{(n)} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Z_{4j-2}^{(n)} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes Z_{4j-1}^{(n)} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes Z_{4j}^{(n)},
\end{aligned}
$$

and for $r + 1 \leq j \leq r + s$ we have that $Z_{4j-3}^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{4n})$, $Z_{4j-2}^{(n)}, Z_{4j-1}^{(n)}, Z_{4j}^{(n)} \in \mathrm{GOE}(n, \frac{1}{4n})$, and

$$
\begin{aligned}
X_j^{(n)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Z_{4j-3}^{(n)} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes (iZ_{4j-2}^{(n)}) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (iZ_{4j-1}^{(n)}) + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \otimes (iZ_{4j}^{(n)}) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Z_{4j-3}^{(n)} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Z_{4j-2}^{(n)} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes Z_{4j-1}^{(n)} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes Z_{4j}^{(n)}.
\end{aligned}
$$

Let $p \in \mathbb{C}\langle X_1, \ldots, X_{r+s} \rangle$. Then there is a polynomial q in $4(r + s)$ non-commuting variables and with coefficients in $M_2(\mathbb{C})$ such that

$$p(X_1^{(n)}, \ldots, X_{r+s}^{(n)}) = q(2Z_1^{(n)}, \ldots, 2Z_{4(r+s)}^{(n)}). \tag{7.1}$$

With $(\mathcal{C}, \phi)$ as in Theorem 7.3, let $\{z_1, \ldots, z_{4(r+s)}\}$ be a semicircular system in $(\mathcal{C}, \phi)$. Set $\mathcal{B} = M_2(\mathbb{C})$, and choose $N \subset \Omega$ as in Theorem 7.3, such that for any $\omega \in \Omega \setminus N$,

$$\lim_{n \to \infty} \|q(2Z_1^{(n)}(\omega), \ldots, 2Z_{4(r+s)}^{(n)}(\omega)))\| = \|q(z_1, \ldots, z_{4(r+s)})\|. \tag{7.2}$$

Clearly, q may be chosen in such a way that

$$q(z_1, \ldots, z_{4(r+s)})$$

$$= p\left[\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes z_{4j-3} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes z_{4j-2} + \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} \otimes z_{4j-1} + \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \otimes z_{4j}\right)_{1 \le j \le r+s}\right]$$

$$= p\left[\left(\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}(z_{4j-3} - z_{4j-2}) & \frac{1}{\sqrt{2}}(z_{4j} + iz_{4j-1}) \\ \frac{1}{\sqrt{2}}(z_{4j} - iz_{4j-1}) & \frac{1}{\sqrt{2}}(z_{4j-3} + z_{4j-2}) \end{pmatrix}\right)_{1 \le j \le r+s}\right].$$

Combining this identity with (7.1) and (7.2) we find that for every $\omega \in \Omega \setminus N$,

$$\lim_{n \to \infty} \|p(X_1^{(n)}(\omega), \ldots, X_{r+s}^{(n)}(\omega))\| = \left\|p\left[\left(\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}(z_{4j-3} - z_{4j-2}) & \frac{1}{\sqrt{2}}(z_{4j} + iz_{4j-1}) \\ \frac{1}{\sqrt{2}}(z_{4j} - iz_{4j-1}) & \frac{1}{\sqrt{2}}(z_{4j-3} + z_{4j-2}) \end{pmatrix}\right)_{1 \le j \le r+s}\right]\right\|$$

and, according to Remark 7.2, this completes the proof. ∎

8 Identifying Λ in special cases.

The aim of this section is to identify the distribution Λ occurring in Lemma 5.5 and Theorem 5.6. We shall concentrate on the cases $S_n \in \text{GOE}(n, \frac{1}{n})$, $S_n \in \text{GOE}^*(n, \frac{1}{n})$, $S_n \in \text{GSE}(n, \frac{1}{n})$ and $S_n \in \text{GSE}^*(n, \frac{1}{n})$.

We let x be a semicircular element in a C^*-probability space $(\mathcal{A}, \tau)$ with τ a faithful state on $\mathcal{A}$.

8.1 Theorem. Let $\phi \in C_c^\infty(\mathbb{R})$. Then

(i) for $n \in \mathbb{N}$ and $X_n \in \text{GOE}(n, \frac{1}{n})$,

$$\mathbb{E}\{\text{tr}_n(\phi(X_n))\} = \tau(\phi(x)) + \frac{1}{2n}\left(\frac{\phi(-2) + \phi(2)}{2} - \frac{1}{\pi}\int_{-2}^2 \frac{\phi(x)}{\sqrt{x^2 - 4}}\,\mathrm{d}x\right) + O\left(\frac{1}{n^2}\right),$$

(8.1)

(ii) for $n \in \mathbb{N}$ and $X_n \in \text{GOE}^*(n, \frac{1}{n})$,

$$\mathbb{E}\{\text{tr}_n(\phi(X_n))\} = \tau(\phi(x)) + \frac{1}{2n}\left(\phi(0) - \frac{1}{\pi}\int_{-2}^2 \frac{\phi(x)}{\sqrt{x^2 - 4}}\,\mathrm{d}x\right) + O\left(\frac{1}{n^2}\right),$$

(8.2)

(iii) for $n \in \mathbb{N}$ and $X_n \in \text{GSE}(n, \frac{1}{n})$,

$$\mathbb{E}\{(\text{tr}_2 \otimes \text{tr}_n)\phi(X_n)\} = \tau(\phi(x)) + \frac{1}{4n}\left(\frac{1}{\pi}\int_{-2}^2 \frac{\phi(x)}{\sqrt{x^2 - 4}}\,\mathrm{d}x - \frac{\phi(-2) + \phi(2)}{2}\right) + O\left(\frac{1}{n^2}\right),$$

(8.3)

and

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(iv) for $n \in \mathbb{N}$ and $X_n \in \text{GSE}^*(n, \frac{1}{n})$,

$$\mathbb{E}\{(\text{tr}_2 \otimes \text{tr}_n)\phi(X_n)\} = \tau(\phi(x)) + \frac{1}{4n}\left(\frac{1}{\pi}\int_{-2}^{2}\frac{\phi(x)}{\sqrt{x^2 - 4}}\,\mathrm{d}x - \phi(0)\right) + O\left(\frac{1}{n^2}\right). \quad (8.4)$$

Proof. (i) Let $X_n \in \text{GOE}(n, \frac{1}{n})$, and for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ put

$$G_n(\lambda) = \mathbb{E}\{\text{tr}_n[(\lambda\mathbf{1}_n - X_n)^{-1}]\}.$$

In this first case corresponding to $m = 1$, $r = 1$, $a_0 = 0$, and $a_1 = \frac{1}{\sqrt{2}}$ we have:

$$\begin{aligned}
R(\lambda) &= \tau[(\lambda\mathbf{1}_{\mathcal{A}} - x)^{-2}] = -g'(\lambda), \\
L(\lambda) &= \tau[(\lambda\mathbf{1}_{\mathcal{A}} - x)^{-1}R(\lambda)g(\lambda)^{-1}(\lambda\mathbf{1}_{\mathcal{A}} - x)^{-1}] \\
&= -R(\lambda)g(\lambda)^{-1}g'(\lambda) \\
&= [g'(\lambda)]^2 g(\lambda)^{-1},
\end{aligned}$$

for $\lambda \in \mathbb{C} \setminus [-2, 2]$.

Then, according to Lemma 5.5,

$$\Lambda(\phi) = \lim_{y \to 0^+} \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}}\phi(x)[L(x + \mathrm{i}y) - L(x - \mathrm{i}y)]\mathrm{d}x, \qquad (\phi \in C_c^\infty(\mathbb{R})) \qquad (8.5)$$

defines a distribution Λ on $\mathbb{R}$ with $\text{supp}(\Lambda) \subseteq [-2, 2]$.

By [VDN, Example 3.4.2]

$$g(\lambda) = \frac{\lambda - \lambda\sqrt{1 - \frac{4}{\lambda^2}}}{2},$$

where $\sqrt{1 - \frac{4}{\lambda^2}}$ means the principal value of $\sqrt{1 - \frac{4}{\lambda^2}}$, i.e. $\sqrt{1 - \frac{4}{\lambda^2}} > 0$ if $\lambda \in \mathbb{R}$, $\lambda > 2$. Hence,

$$g'(\lambda) = \frac{1}{2}\left(1 - \frac{\lambda}{\sqrt{\lambda^2 - 4}}\right),$$

and

$$\begin{aligned}
L(\lambda) &= \frac{1}{4}\left(1 - \frac{\lambda}{\sqrt{\lambda^2 - 4}}\right)^2 \frac{2}{\lambda - \sqrt{\lambda^2 - 4}} \\
&= \frac{1}{2}\left(\frac{\lambda}{\lambda^2 - 4} - \frac{1}{\sqrt{\lambda^2 - 4}}\right).
\end{aligned}$$

With

$$\nu_1 = \frac{1}{2}\left(\delta_{-2} + \delta_2\right), \qquad (8.6)$$

$\frac{\lambda}{\lambda^2 - 4}$ is the Stieltjes transform of ν_1, i.e.

$$G_{\nu_1}(\lambda) = \frac{\lambda}{\lambda^2 - 4}, \qquad (\lambda \in \mathbb{C} \setminus \{\pm 2\}). \qquad (8.7)$$

Moreover, one may calculate the moments of the probability measure ν_2 given by

$$\mathrm{d}\nu_2(x) = \frac{1}{\pi} \frac{1}{\sqrt{x^2 - 4}} \cdot 1_{[-2,2]}(x)\mathrm{d}x \tag{8.8}$$

and use the series expansion of $(\lambda^2 - 4)^{-1/2}$, $(|\lambda| > 2)$, to see that $\frac{1}{\sqrt{\lambda^2-4}}$ is the Stieltjes transform of ν_2, i.e.

$$G_{\nu_2}(\lambda) = \frac{1}{\sqrt{\lambda^2 - 4}}, \qquad (\lambda \in \mathbb{C} \setminus [-2, 2]). \tag{8.9}$$

Altogether

$$L(\lambda) = \frac{1}{2} \left(G_{\nu_1}(\lambda) - G_{\nu_2}(\lambda) \right), \qquad (\lambda \in \mathbb{C} \setminus [-2, 2]). \tag{8.10}$$

Hence, by the inverse Stieltjes transform and (8.5), Λ is the distribution on $\mathbb{R}$ corresponding to the signed measure $\frac{1}{2}(\nu_1 - \nu_2)$.

By Theorem 5.6, for any $\phi \in C_c^\infty(\mathbb{R})$

$$\mathbb{E}\{\mathrm{tr}_n(\phi(X_n))\} = \tau(\phi(x)) + \frac{1}{n}\Lambda(\phi) + O\left(\frac{1}{n^2}\right),$$

that is

$$\mathbb{E}\{\mathrm{tr}_n(\phi(X_n))\} = \tau(\phi(x)) + \frac{1}{2n}\left(\frac{\phi(-2) + \phi(2)}{2} - \frac{1}{\pi}\int_{-2}^{2} \frac{\phi(x)}{\sqrt{x^2 - 4}}\,\mathrm{d}x \right) + O\left(\frac{1}{n^2}\right).$$

(ii) Let $X_n \in \mathrm{GOE}^*(n, \frac{1}{n})$, and for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ put

$$G_n(\lambda) = \mathbb{E}\{\mathrm{tr}_n[(\lambda\mathbf{1}_n - X_n)^{-1}]\}.$$

In this case (corresponding to $m = 1$, $r = 1$, $a_0 = 0$, and $a_1 = \frac{1}{i\sqrt{2}}$),

$$
\begin{aligned}
R(\lambda) &= -\tau[(\lambda\mathbf{1}_\mathcal{A} + x)^{-1}(\lambda\mathbf{1}_\mathcal{A} - x)^{-1}] \\
&= -\frac{\tau[(\lambda\mathbf{1}_\mathcal{A} + x)^{-1} + (\lambda\mathbf{1}_\mathcal{A} - x)^{-1}]}{2\lambda} \\
&= -\frac{\tau[(\lambda\mathbf{1}_\mathcal{A} - x)^{-1}]}{\lambda} \\
&= -\frac{g(\lambda)}{\lambda},
\end{aligned}
$$

where the third equality follows from the fact that x and $-x$ have the same distribution. Thus,

$$
\begin{aligned}
L(\lambda) &= \tau[(\lambda\mathbf{1}_\mathcal{A} - x)^{-1}R(\lambda)g(\lambda)^{-1}(\lambda\mathbf{1}_\mathcal{A} - x)^{-1}] \\
&= -\frac{\tau[(\lambda\mathbf{1}_\mathcal{A} - x)^{-2}]}{\lambda} \\
&= \frac{g'(\lambda)}{\lambda} \\
&= \frac{1}{2}\left(\frac{1}{\lambda} - \frac{1}{\sqrt{\lambda^2 - 4}} \right) \\
&= \frac{1}{2}\left(G_{\nu_3}(\lambda) - G_{\nu_2}(\lambda) \right),
\end{aligned}
$$

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where ν_2 is the measure defined above, and $\nu_3 = \delta_0$.

Hence, for any $\phi \in C_c^\infty(\mathbb{R})$ we have:

$$\mathbb{E}\{\mathrm{tr}_n(\phi(X_n))\} = \tau(\phi(x)) + \frac{1}{2n}\left(\phi(0) - \frac{1}{\pi}\int_{-2}^{2}\frac{\phi(x)}{\sqrt{x^2-4}}\,\mathrm{d}x\right) + O\left(\frac{1}{n^2}\right).$$

(iii) Consider a random matrix $X_n \in \mathrm{GSE}(n, \frac{1}{n})$. For convenience we introduce some notation:

$$\mathcal{J} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and we let $\mathbb{H}^\mathbb{C}$ denote the complexification of the quaternions. Then

$$\mathbb{H}^\mathbb{C} \cong \mathrm{span}_\mathbb{C}\{\mathbf{1}_2, \mathcal{J}, \mathcal{K}, \mathcal{L}\} = M_2(\mathbb{C}).$$

It follows from the definition of $\mathrm{GSE}(n, \frac{1}{n})$ and the relations between $\mathrm{GOE}(n, \frac{1}{n})$, $\mathrm{GOE}^*(n, \frac{1}{n})$ and $\mathrm{GRM}^\mathbb{R}(n, \frac{1}{n})$ mentioned in the Introduction, that there are independent random matrices $Y_1^{(n)}, Y_2^{(n)}, Y_3^{(n)}$ and $Y_4^{(n)}$ from $\mathrm{GRM}^\mathbb{R}(n, \frac{1}{n})$ such that

$$X_n = \sum_{j=1}^{4}(a_j \otimes Y_j^{(n)} + a_j^* \otimes Y_j^{(n)^*}),$$

where

$$a_1 = \frac{1}{2\sqrt{2}}\,\mathbf{1}_2, \quad a_2 = \frac{1}{2\sqrt{2}}\,\mathcal{J}, \quad a_3 = \frac{1}{2\sqrt{2}}\,\mathcal{K}, \text{ and } a_4 = \frac{1}{2\sqrt{2}}\,\mathcal{L}.$$

Let $(\mathcal{B}, \tau)$ be any C^*-probability space with τ a faithful state on $\mathcal{B}$, and let $\{y_1, y_2, y_3, y_4\}$ be a circular system in $(\mathcal{B}, \tau)$. Define $s \in \mathbb{H}^\mathbb{C} \otimes \mathcal{B}$ by

$$s = \sum_{j=1}^{4}(a_j \otimes y_j + a_j^* \otimes y_j^*).$$

Then

$$s = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(y_1+y_1^*) + \frac{i}{\sqrt{2}}(y_2-y_2^*)\right) & \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(y_3-y_3^*) + \frac{i}{\sqrt{2}}(y_4-y_4^*)\right) \\ \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(y_3^*-y_3) + \frac{i}{\sqrt{2}}(y_4-y_4^*)\right) & \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(y_1+y_1^*) + \frac{i}{\sqrt{2}}(y_2^*-y_2)\right) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}(x_1+x_2) & \frac{1}{\sqrt{2}}(ix_3+x_4) \\ \frac{1}{\sqrt{2}}(-ix_3+x_4) & \frac{1}{\sqrt{2}}(x_1-x_2) \end{pmatrix}$$

for a semicircular system $\{x_1, \ldots, x_4\}$ in $(\mathcal{B}, \tau)$. By [VDN, Proposition 5.1.3] and [VDN, Proposition 5.1.6], this representation of s reveals that s is circular in $(M_2(\mathcal{B}), \mathrm{tr}_2 \otimes \tau)$. Hence, it suffices to prove that

$$\mathbb{E}\{(\mathrm{tr}_2 \otimes \mathrm{tr}_n)\phi(X_n)\} = (\mathrm{tr}_2 \otimes \tau)\phi(s) + \frac{1}{4n}\left(\frac{1}{\pi}\int_{-2}^{2}\frac{\phi(x)}{\sqrt{x^2-4}}\,\mathrm{d}x - \frac{\phi(-2)+\phi(2)}{2}\right) + O\left(\frac{1}{n^2}\right).$$

$$(8.11)$$

According to Theorem 5.6 there is a distribution Λ with $\mathrm{supp}(\Lambda) \subseteq \sigma(s) = [-2, 2]$, such that

$$\mathbb{E}\{(\mathrm{tr}_2 \otimes \mathrm{tr}_n)\phi(X_n)\} = (\mathrm{tr}_2 \otimes \tau)\phi(s) + \frac{1}{n}\Lambda(\phi) + O\left(\frac{1}{n^2}\right),$$

As in the two first cases, to identify Λ we try to recognize $l(\lambda) = \mathrm{tr}_2(L(\lambda \mathbf{1}_2))$ as the Stieltjes transform of a signed measure and then apply the inverse Stieltjes transform. By definition

$$l(\lambda) = (\mathrm{tr}_2 \otimes \tau)[(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - s)^{-1}(R(\lambda \mathbf{1}_2)G(\lambda \mathbf{1}_2)^{-1} \otimes \mathbf{1}_\mathcal{B})(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - s)^{-1}], \quad (8.12)$$

where (note that in this case $a_j^* = \pm a_j$)

$$R(\lambda \mathbf{1}_2) = 2\sum_{j=1}^{4}\sum_{k,l=1}^{2}(\mathrm{id}_2 \otimes \tau)[(a_j e_{kl}^{(2)} \otimes \mathbf{1}_\mathcal{B})(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - \overline{s})^{-1}(e_{kl}^{(2)}a_j \otimes \mathbf{1}_\mathcal{B})(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - s)^{-1}].$$

We prove that

$$R(\lambda \mathbf{1}_2) = -\frac{1}{2}\,(\mathrm{id}_2 \otimes \tau)[(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - s)^{-2}] = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}G(\lambda \mathbf{1}_2). \quad (8.13)$$

To this end define a linear map $\Psi : \mathbb{H}^\mathbb{C} \otimes \mathcal{B} \to \mathbb{H}^\mathbb{C} \otimes \mathcal{B}$ by

$$\begin{aligned}
\Psi(\mathbf{1}_2 \otimes b) &= \mathbf{1}_2 \otimes b, \\
\Psi(\mathcal{J} \otimes b) &= -\mathcal{J} \otimes b, \\
\Psi(\mathcal{K} \otimes b) &= \mathcal{K} \otimes b, \\
\Psi(\mathcal{L} \otimes b) &= -\mathcal{L} \otimes b,
\end{aligned}$$

for $b \in \mathcal{B}$. One easily checks that Ψ is actually a (well-defined) unital $*$-isomorphism. In particular, as

$$\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - \overline{s} = \Psi(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - s),$$

we have that

$$(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - \overline{s})^{-1} = (\Psi(\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - s))^{-1} = \Psi((\lambda \mathbf{1}_2 \otimes \mathbf{1}_\mathcal{B} - s)^{-1}). \quad (8.14)$$

Let $x \otimes v$ be an elementary tensor in $\mathbb{H}^\mathbb{C} \otimes \mathcal{B}$. Then

$$\sum_{k,l=1}^{2}(e_{kl}^{(2)} \otimes \mathbf{1}_\mathcal{B})\Psi(x \otimes v)(e_{kl}^{(2)} \otimes \mathbf{1}_\mathcal{B}) = \begin{cases} x \otimes v, & \text{if } x = \mathbf{1}_2, \\ -x \otimes v, & \text{if } x = \mathcal{J}, \\ -x \otimes v, & \text{if } x = \mathcal{K}, \\ -x \otimes v, & \text{if } x = \mathcal{L}, \end{cases}$$

and it is standard to check that this implies that for $x \in \{\mathbf{1}_2, \mathcal{J}, \mathcal{K}, \mathcal{L}\}$ (and hence for any $x \in \mathbb{H}^\mathbb{C}$) we have:

$$\sum_{j=1}^{4}\sum_{k,l=1}^{2}(a_j e_{kl}^{(2)} \otimes \mathbf{1}_\mathcal{B})\Psi(x \otimes v)(e_{kl}^{(2)}a_j \otimes \mathbf{1}_\mathcal{B}) = -\frac{1}{4}(x \otimes v). \quad (8.15)$$

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As $\lambda\mathbf{1}_2 \otimes \mathbf{1}_{\mathcal{B}} - \overline{s}$ belongs to the unital C^*-algebra $\mathbb{H}^{\mathbb{C}} \otimes \mathcal{B}$, $(\lambda\mathbf{1}_2 \otimes \mathbf{1}_{\mathcal{B}} - \overline{s})^{-1}$ also belongs to $\mathbb{H}^{\mathbb{C}} \otimes \mathcal{B}$. Thus, (8.14) and (8.15) imply that

$$\sum_{j=1}^{4} \sum_{k,l=1}^{2} (a_j e_{kl}^{(2)} \otimes \mathbf{1}_{\mathcal{B}})(\lambda\mathbf{1}_2 \otimes \mathbf{1}_{\mathcal{B}} - \overline{s})^{-1}(e_{kl}^{(2)} a_j \otimes \mathbf{1}_{\mathcal{B}}) = -\frac{1}{4}(\lambda\mathbf{1}_2 \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1},$$

and consequently,

$$R(\lambda\mathbf{1}_2) = -\frac{1}{2}(\mathrm{id}_2 \otimes \tau)[(\lambda\mathbf{1}_2 \otimes \mathbf{1}_{\mathcal{B}} - s)^{-2}] = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}G(\lambda\mathbf{1}_2).$$

The next step is to prove that $G(\lambda\mathbf{1}_2) \in \mathbb{C}\mathbf{1}_2$. We have seen that

$$G(\lambda\mathbf{1}_2)^{-1} = \lambda\mathbf{1}_2 - \sum_{j=1}^{4}(a_j G(\lambda\mathbf{1}_2)a_j^* + a_j^* G(\lambda\mathbf{1}_2)a_j),$$

that is

$$G(\lambda\mathbf{1}_2)^{-1} = \lambda\mathbf{1}_2 - \frac{1}{4}(G(\lambda\mathbf{1}_2) - \mathcal{J}G(\lambda\mathbf{1}_2)\mathcal{J} - \mathcal{K}G(\lambda\mathbf{1}_2)\mathcal{K} - \mathcal{L}G(\lambda\mathbf{1}_2)\mathcal{L}). \tag{8.16}$$

Now, for any $x \in M_2(\mathbb{C})$ we have:

$$x - \mathcal{J}x\mathcal{J} - \mathcal{K}x\mathcal{K} - \mathcal{L}x\mathcal{L} = 4\,\mathrm{tr}_2(x)\mathbf{1}_2 \in \mathbb{C}\mathbf{1}_2, \tag{8.17}$$

and therefore, by (8.16), $G(\lambda\mathbf{1}_2)^{-1} \in \mathbb{C}\mathbf{1}_2$. Then $G(\lambda\mathbf{1}_2)$ is a scalar too, so

$$G(\lambda\mathbf{1}_2) = \mathrm{tr}_2(G(\lambda\mathbf{1}_2))\,\mathbf{1}_2 = g(\lambda)\,\mathbf{1}_2,$$

and

$$G(\lambda\mathbf{1}_2)^{-1} = g(\lambda)^{-1}\,\mathbf{1}_2.$$

By (8.13)

$$R(\lambda\mathbf{1}_2) = \frac{1}{2}\,g'(\lambda)\,\mathbf{1}_2,$$

and again, since s is semicircular,

$$g(\lambda) = \frac{\lambda - \lambda\sqrt{1 - \frac{4}{\lambda^2}}}{2}.$$

Inserting these expressions into (8.12) we find that for $\lambda \in \mathbb{C} \setminus [-2, 2]$ we have:

$$\begin{aligned}
l(\lambda) &= \frac{1}{2}\,g'(\lambda)\,g(\lambda)^{-1}\,(\mathrm{tr}_2 \otimes \tau)[(\lambda\mathbf{1}_2 \otimes \mathbf{1}_{\mathcal{B}} - s)^{-2}] \\
&= -\frac{1}{2}\,[g'(\lambda)]^2\,g(\lambda)^{-1} \\
&= -\frac{1}{2}\frac{1}{4}\left(1 - \frac{\lambda}{\sqrt{\lambda^2 - 4}}\right)^2 \frac{2}{\lambda - \sqrt{\lambda^2 - 4}} \\
&= \frac{1}{4}\left(\frac{1}{\sqrt{\lambda^2 - 4}} - \frac{\lambda}{\lambda^2 - 4}\right) \\
&= \frac{1}{4}(G_{\nu_2}(\lambda) - G_{\nu_1}(\lambda)),
\end{aligned}$$

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where ν_1 and ν_2 are the measures defined above. Then (8.11) follows as in the previous cases.

(iv) The proof of (iv) is similar to the proof of (ii), but one must apply some of the techniques from the proof of (iii) too. We leave out the details. ∎

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